

The average number of critical rank-one-approximations to a symmetric tensor

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ABSTRACT. Given a real symmetric tensor $v \in (\mathbb{R}^n)^{\otimes p}$ of order p , a *critical rank-one approximation* of v is a local minimum of the euclidean distance from the set of symmetric rank-1 tensors $\mathcal{V}_{n,p}$ to v . We compute the expected number of critical rank-one-approximations to a random tensor v , that is drawn from the standard Gaussian distribution relative to the Bombieri norm. This answers a question posed by Draisma and Horobet [7], who asked for a closed formula of this expectation. The computation requires to compute the expected absolute value of the determinant of a matrix from the Gaussian Orthogonal Ensemble.

1. INTRODUCTION

Let $S^p\mathbb{R}^n \subset (\mathbb{R}^n)^{\otimes p}$ denote the space of order p symmetric tensors of format $n \times \dots \times n$ and let $\mathcal{V}_{n,p} \subset S^p\mathbb{R}^n$ denote the set of symmetric rank-1 tensors, also called the *Veronese variety*. In application oriented areas such as blind source separation, data compression, imaging or genomic data analysis [10, 13, 14, 22] one is interested in finding the *best symmetric rank-one approximation* to a given $v \in S^p\mathbb{R}^n$; that is, to solve the optimization problem

$$(1.1) \quad \operatorname{argmin}_{x \in \mathcal{V}_{n,p}} \|x - v\|^2$$

Draisma and Horobet [7] pointed out that many algorithms to compute best rank-one approximations compute local optima. They conclude that an appropriate measure of complexity of solving the optimization problem (1.1) is the number of critical points (as elements in projective space) of the function

$$d_v : \mathcal{V}_{n,p} \rightarrow \mathbb{R}, x \mapsto \|x - v\|^2 := \sum_{1 \leq i_1, \dots, i_p \leq n} (v_{i_1, \dots, i_p} - x_{i_1, \dots, i_p})^2.$$

Note that d_v is an algebraic function in the entries of v . When counting also complex critical points (i.e., points where the gradient of d_v vanishes), the number of critical points of d_v is constant on a dense subset of $S^p\mathbb{R}^n$. This number is called the *euclidean distance degree* of $\mathcal{V}_{n,p}$ [8] and it is equal to $D(n, p) := \sum_{i=0}^{n-1} (p-1)^i$ [6].

In applications, however, one often is interested in the number of critical points that are real. Henceforth, we call the number of real critical points of d_v the *real euclidean distance degree of v with respect to $\mathcal{V}_{n,p}$* . What makes things hard is that, in contrast to the complex count, the real euclidean distance degree is not generically constant. For the Veronese variety this can be observed for instance in [18, Theorem 2.3] or in the experimental data from Section 1.3 below. Nevertheless, this motivated Draisma and Horobet [7] to study the *average* number of real critical points of d_v , when v is random. Their model of randomness is a symmetric tensor $v = (v_{i_1, \dots, i_p})$, where the v_{i_1, \dots, i_p} are centered gaussian random variables with variance $\sigma^2 = \left(\frac{p!}{\alpha_1! \dots \alpha_n!}\right)^{-1}$ and α_j is the number of j 's appearing in (i_1, \dots, i_p) . Let us call such a random tensor a *gaussian symmetric tensor*. This distribution is also called standard Gaussian with respect to the Bombieri norm, because the coefficients of a gaussian symmetric tensor v in the Bombieri norm are standard normal gaussian random variables, see [7, Sec. 4.1].

The purpose of this article is to compute *expected real euclidean distance degree* of $\mathcal{V}_{n,p}$ defined as

$$(1.2) \quad E(n, p) := \mathbb{E}_{v \in S^p\mathbb{R}^n \text{ gaussian symmetric}} [\text{real euclidean distance degree of } v \text{ w.r.t. } \mathcal{V}_{n,p}]$$

Remark. In [8] $E(n, p)$ is denoted $\text{aEDdegree}(\mathcal{V}_{n,p})$.

The *Gaussian Orthogonal Ensemble* is a parametric family of random symmetric matrices $A \in \mathbb{R}^{n \times n}$ with probability density functions $\sqrt{2}^{-n} \sqrt{\pi}^{-\frac{n(n+1)}{2}} \exp(-\frac{1}{2\sigma^2} \text{Trace}[(A+uI)^2])$, parametrized by $(u, \sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}$;

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see [15, Sec. 2.3] or [21, Sec. 2.2.6]. We call such a random A a *GOE matrix* and write $A \sim \text{GOE}(n; u, \sigma^2)$. We call the probability distribution imposed by $u = 1$ and $\sigma^2 = 1$ the *standard Gaussian Orthogonal Ensemble* and abbreviate $\text{GOE}(n)$ in this case.

Note that Draisma and Horobet's model of a random tensor is a direct generalization of the standard Gaussian Orthogonal Ensemble from matrices to tensors. It is therefore not surprising that random matrix theory of the Gaussian Orthogonal Ensemble will be of use when computing (1.2). Indeed, in [7, Theorem 4.3] Draisma and Horobet present a formula for $E(n, p)$ in terms of the *expected modulus of the characteristic polynomial of a GOE-matrix*, which is

$$(1.3) \quad E(n, p) = \frac{\sqrt{\pi}}{\sqrt{2}^{n-1} \Gamma(\frac{n}{2})} \mathbb{E}_{\substack{w \sim N(0,1) \\ A \sim \text{GOE}(n-1)}} \left| \det \left(\sqrt{p} w I_{n-1} - \sqrt{2(p-1)} A \right) \right|.$$

Observe that

$$(1.4) \quad \mathbb{E}_{A \sim \text{GOE}(n-1)} \left| \det \left(\sqrt{p} w I_{n-1} - \sqrt{2(p-1)} A \right) \right| = \sqrt{p-1}^{n-1} \mathbb{E}_{A \sim \text{GOE}(n-1; u, 1)} |\det(A)|,$$

where $u = \sqrt{\frac{p}{2(p-1)}} w$. Starting from this, the computation of $E(n, p)$ is divided into two steps: We first compute $\mathbb{E}_{A \sim \text{GOE}(n-1; u, 1)} |\det(A)|$ and then take the expectation over u .

Thus, this article makes a contribution to random tensor theory by computing $E(n, d)$ and also makes a contribution to random matrix theory by giving a formula for $\mathbb{E}_{A \sim \text{GOE}(n; u, 1)} |\det(A)|$ in Theorem 1.1. This result is new to our best knowledge.

In what follows let us denote

$$(1.5) \quad \mathcal{I}_n(u) := \mathbb{E}_{A \sim \text{GOE}(n; u, 1)} |\det(A)|, \quad \text{and} \quad \mathcal{J}_n(u) := \mathbb{E}_{A \sim \text{GOE}(n; u, 1)} \det(A).$$

We remark that $|\mathcal{J}_n(u)| \leq \mathcal{I}_n(u)$ by the triangle inequality. A computation of $\mathcal{J}_n(u)$ can be found in [15, Sec. 22] and the ideas in this paper are very much inspired by this reference. In fact, $\mathcal{I}_n(u)$ can be expressed in terms of $\mathcal{J}_n(u)$ and a collection of *Hermite polynomials*. The following is our first main contribution.

Theorem 1.1 (The expected absolute value of the determinant of a GOE matrix). *Let $u \in \mathbb{R}$ be fixed. Define the functions $P_{-1}(x), P_0(x), P_1(x), P_2(x), \dots$ via*

$$P_k(x) = \begin{cases} -e^{\frac{x^2}{2}} \int_{t=-\infty}^x e^{-\frac{t^2}{2}} dt, & \text{if } k = -1 \\ H_{e_k}(x), & \text{if } k = 0, 1, 2, \dots \end{cases}$$

where $H_{e_k}(x)$ is the k -th (probabilist's) Hermite polynomial; see (2.16). The following holds.

(1) If $n = 2m$ is even, we have

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) + \frac{\sqrt{2\pi} e^{-\frac{u^2}{2}}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \sum_{1 \leq i, j \leq m} \det(\Gamma_1^{i,j}) \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix},$$

where $\Gamma_1^{i,j} := [\Gamma(r + s - \frac{1}{2})]_{\substack{1 \leq r \leq m, r \neq i \\ 1 \leq s \leq m, s \neq j}}$.

(2) If $n = 2m - 1$ is odd we have

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) + \frac{\sqrt{2} e^{-\frac{u^2}{2}}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix},$$

where $\Gamma_2^{i,j} = [\Gamma(r + s + \frac{1}{2})]_{\substack{0 \leq r \leq m-1, r \neq i \\ 0 \leq s \leq m-1, s \neq j}}$.

Theorem 1.1 enables us to compute $E(n, p)$. The proof of the following theorem is given in Section 5.

Theorem 1.2 (The expected real euclidean distance degree of $\mathcal{V}_{n,p}$). We denote by $F(a, b, c, x)$ Gauss' hypergeometric function (see (2.2)) and define the matrices $\Gamma_1^{i,j}$ and $\Gamma_2^{i,j}$ as in Theorem 1.1. For all integers $p \geq 2$ the following holds.

(1) If $n = 2m + 1$, we have

$$E(n, p) = 1 + \frac{\sqrt{\pi} \sqrt{p-1}^{n-2} \sqrt{3p-2}}{\prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)} \sum_{1 \leq i, j \leq m} \frac{\det(\Gamma_1^{i,j}) \Gamma\left(i + j - \frac{1}{2}\right)}{\frac{3-2i-2j}{1-2i+2j} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j-1}} F\left(2-2i, 1-2j, \frac{5}{2} - i - j, \frac{3p-2}{4(p-1)}\right)$$

(2) If $n = 2m$, we have

$$E(n, p) = \frac{\sqrt{p-1}^{n-2} \sqrt{3p-2}}{\prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)} \sum_{j=0}^{m-1} \left[\frac{\sqrt{\pi} \det(\Gamma_2^{0,j}) (2j+1)!}{(-1)^j 2^{2j} j!} \frac{(p-2)^j p}{(p-1)^j (3p-2)} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right) \right. \\ \left. - \frac{\det(\Gamma_2^{0,j}) \Gamma\left(j + \frac{1}{2}\right)}{2 \left(-\frac{3p-2}{4(p-1)}\right)^{j+1}} + \sum_{i=1}^{m-1} \frac{\det(\Gamma_2^{i,j}) \Gamma\left(i + j + \frac{1}{2}\right)}{\frac{(1-2i-2j)}{(1-2i+2j)} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j}} F\left(-2j, -2i+1, \frac{3}{2} - i - j, \frac{3p-2}{4(p-1)}\right) \right].$$

Remark. (1) From the description (2.2) of $F(a, b, c, x)$ it is easy to see that, if both of the numeratorial parameters a, b are non-positive integers, then $F(a, b, c, x)$ is a polynomial in x of degree $\min\{-a, -b\}$. Hence, there exists some $f(x) \in \mathbb{R}[x]$, $\deg(f) = 2m - 1$, with

$$E(2m+1, p) = 1 + \sqrt{(p-1)(3p-2)} (p-1)^{m-1} f\left(\frac{4(p-1)}{3p-2}\right),$$

and there exist $g(x) \in \mathbb{R}[x]$, $\deg(g) = 2m - 2$, and $g_j(x) \in \mathbb{R}[x]$, $\deg(g_j) = j$, such that

$$E(2m, p) = \frac{p(p-1)^{m-1}}{\sqrt{3p-2}} \sum_{j=0}^{m-1} \left(\frac{p-2}{p-1}\right)^j g_j\left(\frac{p^2}{(3p-2)(p-2)}\right) + (p-1)^{m-1} \sqrt{3p-2} g\left(\frac{4(p-1)}{3p-2}\right).$$

Compare also the table in Section 1.1.

- (2) The formula in Theorem 1.2 (2) also holds for $p = 2$: Although $F(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)})$ has a pole of order j at $p = 2$, multiplication with $(p-2)^j$ removes the singularity.
- (3) For all $k \in \mathbb{N}$ we have $\Gamma(k + \frac{1}{2}) = q\sqrt{\pi}$ for some $q \in \mathbb{Q}$; see [19, 43:4:3]. A simple count reveals that that in both formulas of $E(n, p)$ the number of $\sqrt{\pi}$'s in the numerator equals the number of $\sqrt{\pi}$'s in the denominator. This implies that $E(2m+1, p) \in \mathbb{Q}(\sqrt{(p-1)(3p-2)})$ and $E(2m, p) \in \mathbb{Q}(\sqrt{3p-2})$.

1.1. Formulas for $E(n, p)$ for small values of n . Using the formulas in Theorem 1.2 we can compute $E(n, p)$ for some small values of n . We used SAGE [20] to compute it for $2 \leq n \leq 9$, the formulas are presented in table 1 below. The source code of the scripts are given in appendix A.

n	$E(n, p)$
2	$\sqrt{3p-2}$
3	$1 + \frac{4(p-1)^{\frac{3}{2}}}{\sqrt{3p-2}}$
4	$\frac{29p^3 - 63p^2 + 48p - 12}{2(3p-2)^{\frac{3}{2}}}$
5	$1 + \frac{2(5p-2)^2(p-1)^{\frac{5}{2}}}{(3p-2)^{\frac{5}{2}}}$
6	$\frac{1339p^6 - 5946p^5 + 11175p^4 - 11240p^3 + 6360p^2 - 1920p + 240}{8(3p-2)^{\frac{7}{2}}}$
7	$1 + \frac{(1099p^4 - 2296p^3 + 2184p^2 - 992p + 176)(p-1)^{\frac{7}{2}}}{2(3p-2)^{\frac{9}{2}}}$
8	$\frac{28473p^9 - 191985p^8 + 579279p^7 - 1022091p^6 + 1160040p^5 - 877380p^4 + 441840p^3 - 142800p^2 + 26880p - 2240}{16(3p-2)^{\frac{11}{2}}}$
9	$1 + \frac{(22821p^6 - 77580p^5 + 118476p^4 - 95136p^3 + 41904p^2 - 9408p + 832)(p-1)^{\frac{9}{2}}}{4(3p-2)^{\frac{13}{2}}}$

TABLE 1. Formulas for $E(n, p)$ for $2 \leq n \leq 9$.

1.2. Comparison to prior results. The critical points of d_v for $v \in S^p(\mathbb{R}^n)$ are sometimes called the *eigenvectors* of v . The reason for this is as follows. We can interpret v as a multilinear map from $(\mathbb{C}^n)^p \rightarrow \mathbb{C}$. Let e_1, \dots, e_n denote the standard basis vectors in \mathbb{C}^n . For a general tensor $v \in (\mathbb{R}^n)^{\otimes d}$ (not necessarily symmetric) eigenpairs of v are pairs $(x, t) \in (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}$ that are defined by the equation

$$(1.6) \quad vx^{p-1} := \begin{pmatrix} v(x, \dots, x, e_1) \\ \vdots \\ v(x, \dots, x, e_n) \end{pmatrix} = tx.$$

Note that (x, t) is an eigenpair of v if and only if $(ax, a^{p-2}t)$, $a \in \mathbb{C}^\times$, is an eigenpair of v . Cartwright and Sturmfels [6] call eigenpairs related by this scaling procedure *equivalent*. The discussion in [9, Sec. 6] reveals that for any $v \in S^p(\mathbb{R}^n)$ and x an eigenvector of v , the symmetric rank-one tensor $x^{\otimes p}$ is a critical point of d_v . Therefore, the number of real critical points of v equals the number of equivalence classes of real eigenpairs of v .

We may compare the results in this article with [5], where we computed the expected number of equivalence classes of real eigenpairs of a random tensor $v = (v_{i_1, \dots, i_p})$, where the v_{i_1, \dots, i_p} are centered random variables with variance $\sigma^2 = 1$. While gaussian symmetric tensors are a generalization of the Gaussian Orthogonal Ensemble, this model of a random tensor is a generalization of the *real Ginibre Ensemble* [11] from matrices to tensors. Let us denote the expected number of real eigenpairs of such a random tensor v in analogy to (1.2) by $E_{\text{non-sym}}(n, p)$. In table 2 we give $D(4, p)$, $E(4, p)$ and $E_{\text{non-sym}}(4, p)$ for $2 \leq p \leq 10$ rounded to 10^{-2} . Compare [7, Sec. 5.2].

p	2	3	4	5	6	7	8	9	10
$D(4, p) \approx$	4	15	40	85	156	259	400	585	820
$E(4, p) \approx$	4	9.4	16.26	24.31	33.38	43.38	54.22	65.84	78.19
$E_{\text{non-sym}}(4, p) \approx$	1.95	3.85	6.32	9.22	12.49	16.10	20.00	24.19	28.64

TABLE 2. Table of values of $D(4, p)$, $E(4, p)$ and $E_{\text{non-sym}}(4, p)$ for $2 \leq p \leq 10$.

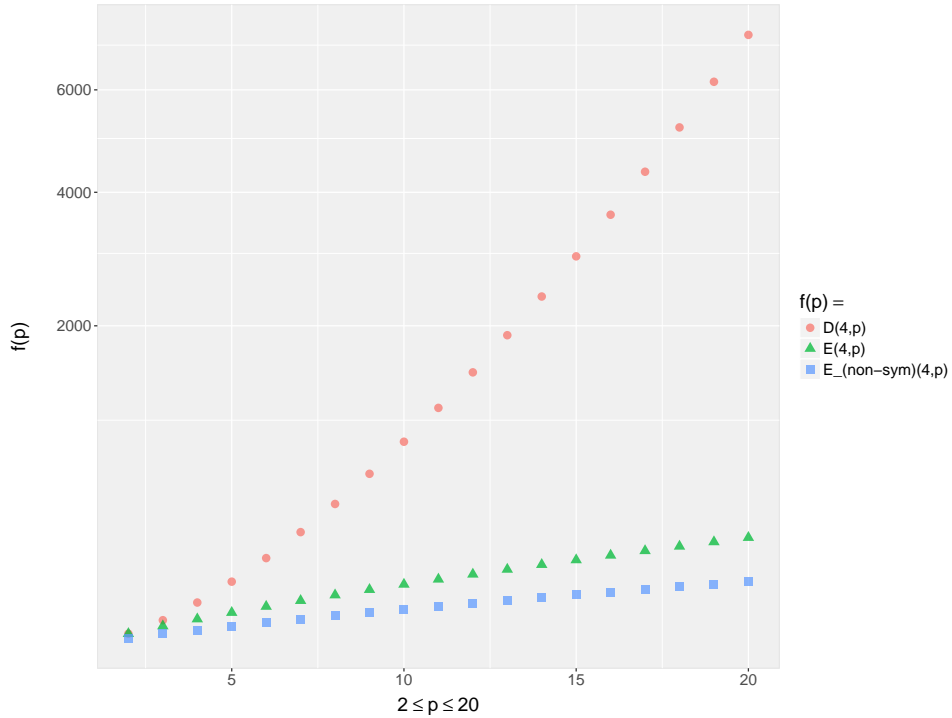


FIGURE 1.1. Plot of $p \mapsto f(p)$ for $f(p) \in \{D(4, p), E(4, p), E_{\text{non-sym}}(4, p)\}$ for the values $2 \leq p \leq 20$. The y -axis is scaled by taking the square root. The plot was created using the `ggplot` package in R [17].

1.3. Experiments. To compare the theoretical results of Theorem 1.2 with actual data we made the following experiment: For $(n, p) \in \{(4, 3), (5, 3), (4, 4), (5, 4)\}$ we sampled 2000 gaussian symmetric tensors in $(\mathbb{R}^n)^{\otimes p}$ using the `rnorm()` command in R [17]. For each of those tensors we computed the real euclidean distance degree by calling BERTINI [3] to compute the number of real eigenpairs of the tensor. The result of the experiments and histograms of the data are given below. We abbreviate 'real euclidean distance degree' by 'r.e.d.d'.

$n = 4, p = 3$									
r.e.d.d	1	3	5	7	9	11	13	15	
count	15	64	143	352	553	553	257	63	
									mean $E(4, 3)$
									$\approx 9.37 \approx 9.4$

$n = 5, p = 3$														
r.e.d.d	1	3	5	7	9	11	13	15	17	19	21	23	25	27
count	1	3	21	53	92	195	284	337	396	311	183	83	33	4
														mean $E(5, 3)$
														$\approx 15.82 \approx 15.75$

$n = 4, p = 4$												
r.e.d.d	6	8	10	12	14	16	18	20	22	24	26	28
count	13	57	128	262	316	377	345	239	155	75	26	7
												mean $E(4, 4)$
												$\approx 16.24 \approx 16.25$

$n = 5, p = 4$														
r.e.d.d	11	13	15	17	19	21	23	25	27	29	31	33	35	37
count	1	1	13	17	37	56	112	117	187	212	200	209	197	149
r.e.d.d	39	41	43	45	47	49	51	53	55	57	59			
count	141	108	86	65	43	30	8	5	4	1	1			
												mean $E(5, 4)$		
												$\approx 32.81 \approx 32.94$		

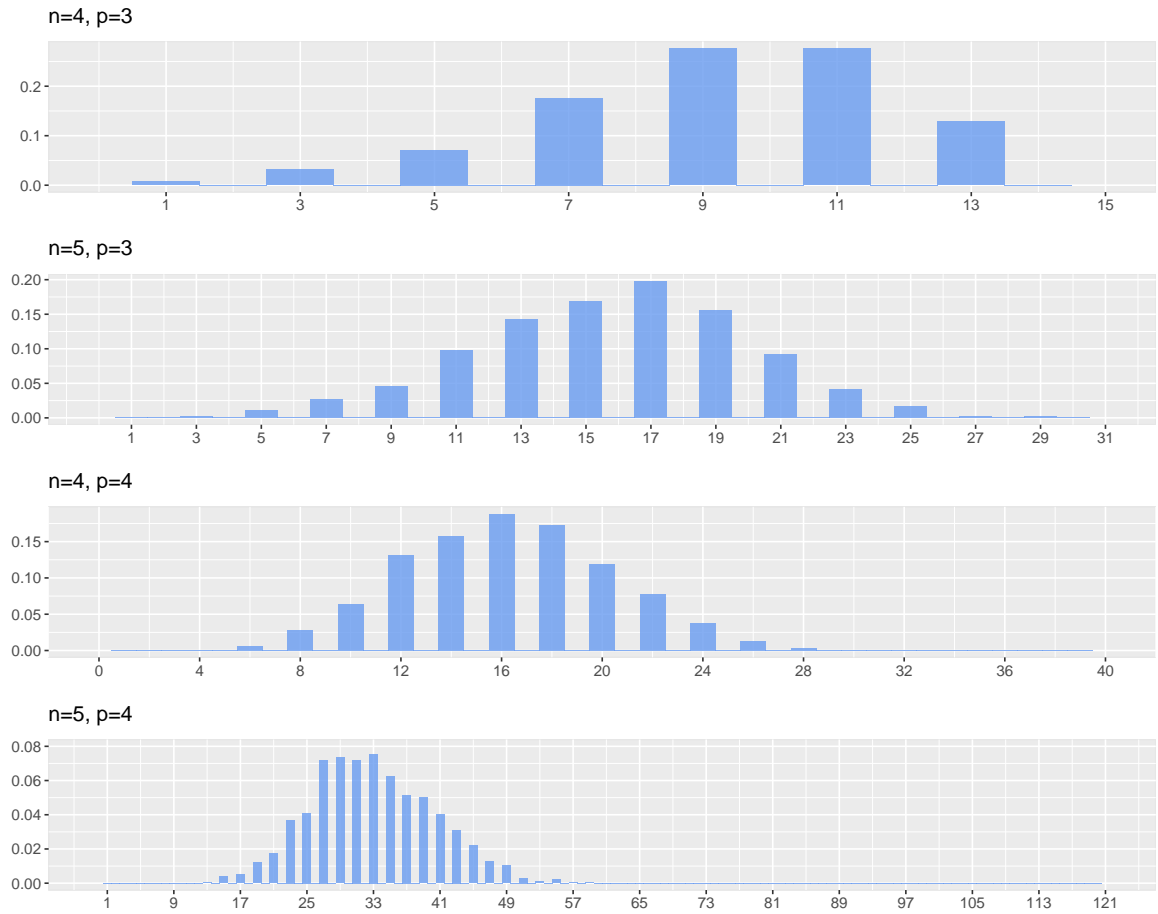


FIGURE 1.2. Histograms of the experimental data. The largest value on the x -axis is the respective complex count $D(n, p) = \sum_{i=0}^{n-1} (p-1)^i$. The y -axis displays the relative frequency of the data. The histograms were created using the `ggplot` package in R [17].

Remark. The numbers in the table $n = 5, p = 3$ add up to only 1999, because BERTINI failed to compute the correct number of real eigenvalues in one case. It falsely computed a double root. We believe that the reason for this happening is that the predictor for double roots in BERTINI, the condition number of the polynomial equation (1.6), may not be appropriate for the problem of solving for tensor eigenpairs, compare the discussion in [4, Sec. 1.2]. Further, note that the distribution in the histogram for $n = 5, p = 4$ looks unlike the others and that there is a gap between $E(5, 4)$ and the corresponding sample mean. We believe that this as well is caused by BERTINI counting the wrong number of real solutions.

1.4. Organization. The rest of this paper is organized as follows. In the next section we will first present some preliminary material that is needed throughout the article. Thereafter, in Section 3 we compute some of the integrals that are needed to prove Theorem 1.1 and Theorem 1.2, which we will then do in Section 4 and Section 5, respectively. Finally, in the appendix we present the SAGE code that we used in Section 1.1.

2. PRELIMINARIES

We first fix notation: In what follows $n \geq 2$ is always a positive integer and $m := \lceil \frac{n}{2} \rceil$. That is $n = 2m$, if n is even, and $n = 2m - 1$, if n is odd. The small letters a, b, c, x, y, λ will denote variables or real numbers. By capital calligraphic letters $\mathcal{A}, \mathcal{M}, \mathcal{K}, \mathcal{L}$ we denote matrices. The symbols G and P are reserved for the functions defined in (2.16) and M and F denote the two hypergeometric functions defined in (2.1) and (2.2) below. The symbol $\langle \cdot, \cdot \rangle$ always denotes the inner product defined in (2.18).

2.1. Special functions. Throughout the article a collection of special function appears. We present them in this subsection. The *Pochhammer polynomials* [19, 18:3:1] are defined by $(x)_n := x(x+1) \dots (x+n-1)$ where n is a positive integer, and $(x)_0 := 1$. *Kummer's confluent hypergeometric function* [19, Sec. 47] is defined as

$$(2.1) \quad M(a, c, x) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!},$$

and *Gauss' hypergeometric function* [19, Sec. 60] is defined as

$$(2.2) \quad F(a, b, c, x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $a, b, c \in \mathbb{R}$, $c \neq 0, -1, -2, \dots$. Generally, neither $M(a, c, x)$ nor $F(a, b, c, x)$ converges for all x . But if either of the numeratorial parameters a, b is a non-positive integer, both $M(a, c, x)$ and $F(a, b, c, x)$ reduce to polynomials and hence are defined for all $x \in \mathbb{R}$ (and this is the only case we will meet throughout the paper).

Remark. Other usual notations are $M(a, c, x) = {}_1F_1(a; c; x)$ and $F(a, b, c, x) = {}_2F_2(a, b; c, x)$. This is due to the fact that both $M(a, c, x)$ and $F(a, b, c, x)$ are special cases of the *general hypergeometric function*, denoted ${}_qF_p(a_1, \dots, a_q; c_1, \dots, c_p; x) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_k}{\prod_{i=1}^q (c_i)_k} \frac{x^k}{k!}$.

The following will be useful.

Lemma 2.1. *Let a, b be non-positive integers and $c \neq 0, -1, -2, \dots$. Then*

$$F(a, b+1, c, x) - F(a+1, b, c, x) = \frac{(a-b)x}{c} F(a+1, b+1, c+1, x)$$

Proof. Since a and b are non-negative integers, $F(a, b+1, c, x)$ and $F(a+1, b, c, x)$ are polynomials, whose constant term is equal to 1. Therefore,

$$(2.3) \quad F(a, b+1, c, x) - F(a+1, b, c, x) = \sum_{k=1}^{\infty} \frac{(a)_k (b+1)_k - (a+1)_k (b)_k}{(c)_k} \frac{x^k}{k!}.$$

We have

$$(a)_k (b+1)_k - (b)_k (a+1)_k \stackrel{[19, 18:5:6]}{=} (a)_k (b)_k \left(1 + \frac{k}{b}\right) - (a)_k (b)_k \left(1 + \frac{k}{a}\right) = (a)_k (b)_k k \frac{a-b}{ab}.$$

According to [19, 18:5:7] the latter is equal to $(a+1)_{k-1}(b+1)_{k-1}k(a-b)$ and, moreover, $(c)_k = c(c+1)_{k-1}$. The claim follows when plugging this into (2.3). \square

For $x \geq 0$ the *gamma function* [19, Sec. 43] is defined as

$$(2.4) \quad \Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt.$$

The *cumulative distribution function of the normal distribution* [19, 40:14:2] and the *error function* [19, 40:3:2] are respectively defined as

$$(2.5) \quad \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{t^2}{2}} dt, \quad \text{and} \quad \text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

The error function and $\Phi(x)$ are related by the following equation [19, 40:14:2]

$$(2.6) \quad 2\Phi(x) = 1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right).$$

The error function and the Kummer's hypergeometric function are related by

$$(2.7) \quad \text{erf}(x) = \frac{2x}{\sqrt{\pi}} M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right);$$

see [1, 13.6.19].

2.2. Hermite polynomials. Hermite polynomials are a family of integer indexed polynomials $H_0(x), H_1(x), \dots$ that are defined via [19, 24:3:2]

$$(2.8) \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

An alternative Hermite function is defined by

$$(2.9) \quad H_{e_n}(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

The two definitions are related by the following equality [19, 24:1:1]

$$(2.10) \quad H_{e_n}(x) = \frac{1}{\sqrt{2}^n} H_n\left(\frac{x}{\sqrt{2}}\right).$$

By [19, 24:5:1] we have that

$$(2.11) \quad H_k(-z) = (-1)^k H_k(z) \quad \text{and} \quad H_{e_k}(-z) = (-1)^k H_{e_k}(z)$$

Remark. In the literature the polynomials $H_n(x)$ are sometimes called the *physicists' Hermite polynomials* and the $H_{e_n}(x)$ are sometimes called the *probabilists' Hermite polynomials*. We will refer to both simply as *Hermite polynomials* and distinguish them by use of the respective symbols.

Hermite polynomials can be expressed in terms of Kummer's confluent hypergeometric function from (2.1):

$$(2.12) \quad H_{2k+1}(x) = (-1)^k \frac{(2k+1)! 2x}{k!} M\left(-k, \frac{3}{2}, x^2\right)$$

$$(2.13) \quad H_{2k}(x) = (-1)^k \frac{(2k)!}{k!} M\left(-k, \frac{1}{2}, x^2\right);$$

see [1, 13.6.17 and 13.6.18].

2.3. Orthogonality relations of the Hermite polynomials. The Hermite polynomials satisfy the following orthogonality relations. Let $\Gamma(x)$ be the Gamma function from (2.4). By [12, 7.374.2] we have

$$(2.14) \quad \int_{\mathbb{R}} H_{e_m}(x) H_{e_n}(x) e^{-x^2} dx = \begin{cases} (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} \Gamma\left(\frac{m+n+1}{2}\right), & \text{if } m+n \text{ is even} \\ 0, & \text{if } m+n \text{ is odd.} \end{cases},$$

and, more generally, if $m + n$ is even, by [2, p. 289, eq. (12)] we have for $\alpha > 0$, $\alpha^2 \neq \frac{1}{2}$ that

$$(2.15) \quad \int_{-\infty}^{\infty} H_{e_m}(x) H_{e_n}(x) e^{-\alpha^2 x^2} dx = \frac{(1 - 2\alpha^2)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right)}{\alpha^{m+n+1}} F\left(-m-n; \frac{1-m-n}{2}; \frac{\alpha^2}{2\alpha^2-1}\right),$$

where $F(a, b, c, x)$ is Gauss' hypergeometric function as defined in (2.2). Recall from (2.5) the definition of $\Phi(x)$. In the following we abbreviate

$$(2.16) \quad P_k(x) := \begin{cases} H_{e_k}(x), & \text{if } k = 0, 1, 2, \dots \\ -\sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(x), & \text{if } k = -1. \end{cases}$$

and put

$$(2.17) \quad G_k(x) := \int_{-\infty}^x P_k(y) e^{-\frac{y^2}{2}} dy, \quad k = 0, 1, 2, \dots$$

We can express the functions $G_k(x)$ in terms of the $P_k(x)$.

Lemma 2.2. *We have*

- (1) For all k : $G_k(x) = -e^{-\frac{x^2}{2}} P_{k-1}(x)$.
- (2) $G_k(\infty) = \begin{cases} \sqrt{2\pi}, & \text{if } k = 0 \\ 0, & \text{if } k \geq 1 \end{cases}$

Proof. Note that (2) is a direct consequence of (1). For (1) let $k \geq 0$ and write

$$G_k(x) = \int_{y=-\infty}^x P_k(y) e^{-\frac{y^2}{2}} dy \stackrel{\text{by (2.16)}}{=} \int_{y=-\infty}^x H_{e_k}(y) e^{-\frac{y^2}{2}} dy \stackrel{\text{by (2.9)}}{=} \int_{y=-\infty}^x (-1)^k \frac{d^k}{dy^k} e^{-\frac{y^2}{2}} dy.$$

Thus $G_k(x) = (-1)^k \frac{d^{k-1}}{dx^{k-1}} e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} P_{k-1}(x)$ as desired. \square

We now fix the following notation: If two functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $\int_{\mathbb{R}} f(x) e^{-x^2} dx < \infty$ and $\int_{\mathbb{R}} g(x) e^{-x^2} dx < \infty$, we define

$$(2.18) \quad \langle f(x), g(x) \rangle := \int_{\mathbb{R}} f(x) g(x) e^{-\frac{x^2}{2}} dx,$$

which is finite due to the Cauchy-Schwartz inequality. The functions $P_k(x)$ and $G_k(x)$ satisfy the following orthogonality relations

Lemma 2.3. *For all $k, \ell \geq 0$ we have*

- (1) $\langle G_k(x), P_\ell \rangle = -\langle G_\ell(x), P_k(x) \rangle$.
- (2) $\langle G_k(x), P_\ell(x) \rangle = \begin{cases} (-1)^{i+j} \Gamma\left(i + j - \frac{1}{2}\right), & \text{if } k = 2i - 1 \text{ and } \ell = 2j \\ 0, & \text{if } k + \ell \text{ is even} \end{cases}$

Proof. For (1) we have

$$\begin{aligned} \langle G_k(x), P_\ell(x) \rangle &= \int_{\mathbb{R}} G_k(x) P_\ell(x) e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \left(\int_{-\infty}^x P_k(y) e^{-\frac{y^2}{2}} dy \right) P_\ell(x) e^{-\frac{x^2}{2}} dx \\ &= \int_{\mathbb{R}} \left(\int_y^\infty P_\ell(x) e^{-\frac{x^2}{2}} dx \right) P_k(y) e^{-\frac{y^2}{2}} dy \\ &= (-1)^\ell \int_{\mathbb{R}} \left(\int_{-\infty}^{-y} P_\ell(x) e^{-\frac{x^2}{2}} dx \right) P_k(y) e^{-\frac{y^2}{2}} dy \\ &= (-1)^{k+\ell} \langle G_\ell(x), P_k(x) \rangle, \end{aligned}$$

where the fourth equality is due to the transformation $x \mapsto -x$ and equation (2.11) and the fifth equality is obtained using the transformation $y \mapsto -y$ and (2.11). This shows (1) for the case $k + \ell$ odd. Further, for $k + \ell$ even we get $\langle P_\ell, P_k \rangle = 0$, which proves (1) and (2) for this case.

Now assume that $k = 2i - 1, \ell = 2j$, in particular $k \neq 0$. We use Lemma 2.2 to write

$$(2.19) \quad \begin{aligned} \langle G_k(x), P_\ell(x) \rangle &= - \int_{\mathbb{R}} P_{k-1}(x) P_\ell(x) e^{-x^2} dx = - \int_{\mathbb{R}} P_{2i-2}(x) P_{2j}(x) e^{-x^2} dx \\ &= - \int_{\mathbb{R}} H_{e_{2i-2}}(x) H_{e_{2j}}(x) e^{-x^2} dx. \end{aligned}$$

By (2.14) we have

$$\int_{\mathbb{R}} H_{e_m}(x) H_{e_n}(x) e^{-x^2} dx = \begin{cases} (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} \Gamma\left(\frac{m+n+1}{2}\right), & \text{if } m+n \text{ is even} \\ 0, & \text{if } m+n \text{ is odd.} \end{cases}$$

Applying this to (2.19) we see that $\langle G_k(x), P_\ell(x) \rangle = (-1)^{i+j} \Gamma\left(i + j - \frac{1}{2}\right)$. This finishes the proof. \square

2.4. The expectation of Hermite polynomials. In this section we will compute the expected value of the Hermite polynomials when the argument follows a normal distribution.

Lemma 2.4. For $\sigma^2 > 0$ we have $\mathbb{E}_{u \sim N(0, \sigma^2)} H_{2k}(u) = \frac{(2k)!}{k!} (2\sigma^2 - 1)^k$.

Proof. Write

$$\mathbb{E}_{u \sim N(0, \sigma^2)} H_{2k}(u) \stackrel{\text{by definition}}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{u=-\infty}^{\infty} H_{2k}(u) e^{-\frac{u^2}{2\sigma^2}} du = \frac{1}{\sqrt{\pi}} \int_{w=-\infty}^{\infty} H_{2k}(\sqrt{2\sigma^2} w) e^{-w^2} dw,$$

where the second equality is due to the change of variables $w := \frac{u}{\sqrt{2\sigma^2}}$. Applying [12, 7.373.2] we get

$$\frac{1}{\sqrt{\pi}} \int_{w=-\infty}^{\infty} H_{2k}(\sqrt{2\sigma^2} w) e^{-w^2} dw = \frac{(2k)! (2\sigma^2 - 1)^k}{k!}.$$

This finishes the proof. \square

Lemma 2.5. Let $\sigma^2 > 0$ and recall from (2.16) the definition of $P_k(x)$, $k = -1, 0, 1, 2, \dots$

(1) If $k, \ell > 0$ and $k + \ell$ is even, we have

$$\mathbb{E}_{u \sim N(0, \sigma^2)} P_k(u) P_\ell(u) e^{-\frac{u^2}{2}} = \frac{(-1)^{\frac{k+\ell}{2}} \sqrt{2}^{k+\ell} \Gamma\left(\frac{k+\ell+1}{2}\right)}{\sqrt{\pi} \sqrt{\sigma^2 + 1}^{k+\ell+1}} F\left(-k, -\ell; \frac{1-k-\ell}{2}; \frac{\sigma^2 + 1}{2}\right).$$

(2) For all k we have

$$\mathbb{E}_{u \sim N(0, \sigma^2)} P_{-1}(u) P_{2k+1}(u) e^{-\frac{u^2}{2}} = \frac{(-1)^{k+1} (2k+1)!}{2^k k!} \frac{(1 - \sigma^2)^k \sigma^2}{\sqrt{1 + \sigma^2}} F\left(-k, \frac{1}{2}, \frac{3}{2}, \frac{\sigma^4}{\sigma^4 - 1}\right).$$

Proof. To prove (1) we write

$$\mathbb{E}_{u \sim N(0, \sigma^2)} P_k(u) P_\ell(u) e^{-\frac{u^2}{2}} = \mathbb{E}_{u \sim N(0, \sigma^2)} H_{e_k}(u) H_{e_\ell}(u) e^{-\frac{u^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{u=-\infty}^{\infty} H_{e_k}(u) H_{e_\ell}(u) e^{-\frac{u^2}{2} \left(1 + \frac{1}{\sigma^2}\right)} du.$$

Put $\alpha^2 := \frac{1}{2} \left(1 + \frac{1}{\sigma^2}\right)$ and observe that $\alpha^2 \neq 1$. By (2.15) we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\sigma^2}} \int_{u=-\infty}^{\infty} H_{e_k}(u) H_{e_\ell}(u) e^{-\frac{u^2}{2} \left(1 + \frac{1}{\sigma^2}\right)} du \\ &= \frac{(1 - 2\alpha^2)^{\frac{k+\ell}{2}} \Gamma\left(\frac{k+\ell+1}{2}\right)}{\sqrt{2\pi\sigma^2} \alpha^{k+\ell+1}} F\left(-k, -\ell; \frac{1-k-\ell}{2}; \frac{\alpha^2}{2\alpha^2 - 1}\right) \\ &= \frac{(-1)^{\frac{k+\ell}{2}} \sqrt{2}^{k+\ell} \Gamma\left(\frac{k+\ell+1}{2}\right)}{\sqrt{\pi} \sqrt{\sigma^2 + 1}^{k+\ell+1}} F\left(-k, -\ell; \frac{1-k-\ell}{2}; \frac{\sigma^2 + 1}{2}\right) \end{aligned}$$

This proves (1). For (2) we have

$$\mathbb{E}_{u \sim N(0, \sigma^2)} P_{-1}(u) P_{2k+1}(u) e^{-\frac{u^2}{2}} = -\sqrt{2\pi} \mathbb{E}_{u \sim N(0, \sigma^2)} \Phi(u) H_{e_{2k+1}}(u) = \frac{-1}{\sigma} \int_{u=-\infty}^{\infty} \Phi(u) H_{e_{2k+1}}(u) e^{-\frac{u^2}{2\sigma^2}} du.$$

Making a change of variables $x := \frac{u}{\sqrt{2}}$ the right-hand integral becomes

$$\frac{-1}{\sqrt{2}\sigma} \int_{u=-\infty}^{\infty} \Phi(\sqrt{2}x) H_{e_{2k+1}}(\sqrt{2}x) e^{-\frac{x^2}{\sigma^2}} dx = \frac{-1}{2^{k+1}\sigma} \int_{x=-\infty}^{\infty} (1 + \operatorname{erf}(x)) H_{2k+1}(x) e^{-\frac{x^2}{\sigma^2}} dx$$

the equality due to (2.6) and (2.10). We know from (2.11) that $H_{2k+1}(x)$ is an odd function, which implies that $\int_{w=-\infty}^{\infty} H_{2k+1}(x) e^{-\frac{x^2}{\sigma^2}} dx = 0$. Moreover, by (2.12) we have $H_{2k+1}(x) = (-1)^k \frac{(2k+1)!}{k!} M(-k, \frac{3}{2}, x^2)$ and by (2.7) we have $\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}} M(\frac{1}{2}, \frac{3}{2}, -x^2)$. All this shows that

$$\begin{aligned} \mathbb{E}_{u \sim N(0, \sigma^2)} P_{-1}(u) P_{2k+1}(u) e^{-\frac{u^2}{2}} &= \frac{(-1)^{k+1} (2k+1)!}{2^{k-1} \sqrt{\pi} \sigma k!} \int_{x=-\infty}^{\infty} x^2 M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) M\left(-k, \frac{3}{2}, x^2\right) e^{-\frac{x^2}{\sigma^2}} dx \\ (2.20) \quad &= \frac{(-1)^{k+1} (2k+1)!}{2^{k-2} \sqrt{\pi} \sigma k!} \int_{x=0}^{\infty} x^2 M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) M\left(-k, \frac{3}{2}, x^2\right) e^{-\frac{x^2}{\sigma^2}} dx. \end{aligned}$$

where for the second equality we used that the integrand is an even function. Making a change of variables $t := x^2$ we see that

$$(2.21) \quad \int_{x=0}^{\infty} x^2 M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) M\left(-k, \frac{3}{2}, x^2\right) e^{-\frac{x^2}{\sigma^2}} dx = \frac{1}{2} \int_{t=0}^{\infty} \sqrt{t} M\left(\frac{1}{2}, \frac{3}{2}, -t\right) M\left(-k, \frac{3}{2}, t\right) e^{-\frac{t}{\sigma^2}} dt.$$

By [12, 7.622.1] we have

$$(2.22) \quad \int_{t=0}^{\infty} \sqrt{t} M\left(\frac{1}{2}, \frac{3}{2}, -t\right) M\left(-k, \frac{3}{2}, t\right) e^{-\frac{t}{\sigma^2}} dt = \Gamma\left(\frac{3}{2}\right) \frac{(1-\sigma^2)^k \sigma^3}{\sqrt{1+\sigma^2}} F\left(-k, \frac{1}{2}, \frac{3}{2}, \frac{\sigma^4}{\sigma^4-1}\right)$$

Plugging (2.22) into (2.21) and the result into (2.20) we obtain

$$\begin{aligned} \mathbb{E}_{u \sim N(0, \sigma^2)} P_{-1}(u) P_{2k+1}(u) e^{-\frac{u^2}{2}} &= \frac{(-1)^{k+1} (2k+1)!}{2^{k-1} \sqrt{\pi} \sigma k!} \Gamma\left(\frac{3}{2}\right) \frac{(1-\sigma^2)^k \sigma^3}{\sqrt{1+\sigma^2}} F\left(-k, \frac{1}{2}, \frac{3}{2}, \frac{\sigma^4}{\sigma^4-1}\right) \\ &= \frac{(-1)^{k+1} (2k+1)!}{2^k k!} \frac{(1-\sigma^2)^k \sigma^2}{\sqrt{1+\sigma^2}} F\left(-k, \frac{1}{2}, \frac{3}{2}, \frac{\sigma^4}{\sigma^4-1}\right). \end{aligned}$$

For the second equality we have used that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, see [19, 43:4:3]. This finishes the proof. \square

3. COMPUTATION OF THE INTEGRALS THAT APPEAR

This section is dedicated to the computation of the integrals that appear in the subsequent sections. We start with a general lemma that we will later apply to the equations (4.9) and (4.13).

Lemma 3.1. *Let $f : \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}$, $((x_1, \dots, x_m), u) \mapsto f(x_1, \dots, x_m, u)$ be a measurable function, such that $\int_{\mathbb{R}^m \times \mathbb{R}^k} f dx_1 \dots dx_m du < \infty$. Assume that f is invariant under any permutations of the x_i . Then*

$$\sum_{j=0}^m \binom{m}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} f(x_1, \dots, x_m, u) dx_1 \dots dx_m = \int_{x_1, \dots, x_m \in \mathbb{R}} f(x_1, \dots, x_m, u) dx_1 \dots dx_m$$

for all $u \in \mathbb{R}^k$.

Proof. We prove the statement by induction. For $m = 1$ we have

$$\int_{x_1 \leq u} f(x_1, u) dx_1 + \int_{u \leq x_1} f(x_1, u) dx_1 = \int_{x_1 \in \mathbb{R}} f(x_1, u) dx_1.$$

For $m > 1$ we write

$$g_j(u) := \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} f(x_1, \dots, x_m, u) \, dx_1 \dots dx_m, \quad j = 0, \dots, m$$

Using $\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$ [19, 6:5:3] we have

$$(3.1) \quad \sum_{j=0}^m \binom{m}{j} g_j(u) = \sum_{j=0}^{m-1} \binom{m-1}{j} (g_j(u) + g_{j+1}(u)),$$

where

$$g_j(u) + g_{j+1}(u) = \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+2}, \dots, x_m}} \int_{x_{j+1}=-\infty}^{\infty} f(x_1, \dots, x_m, u) dx_1 \dots dx_m.$$

By assumption f is invariant under any permutations of the x_i . Thus, making a change of variables that interchanges x_{j+1} and x_m we see that

$$g_j(u) + g_{j+1}(u) = \int_{x_m=-\infty}^{\infty} \left[\int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-1}}} f(x_1, \dots, x_{m-1}, x_m, u) dx_1 \dots dx_{m-1} \right] dx_m.$$

Plugging this into (3.1) and interchanging summation and integration we obtain

$$\sum_{j=0}^m \binom{m}{j} g_j(u) = \int_{x_m=-\infty}^{\infty} \left[\sum_{j=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-1}}} f(x_1, \dots, x_{m-1}, x_m, u) dx_1 \dots dx_{m-1} \right] dx_m.$$

We can now apply the induction hypothesis to the inner integral and conclude the proof. \square

Recall from (2.16) the definition of the polynomials $P_k(x)$ and from (2.17) the definition of $G_k(x)$. The following lemma is the key to prove Proposition 3.3 and Proposition 3.4 below.

Lemma 3.2. *Let \mathcal{A} denote the $2m \times 2(m-1)$ matrix*

$$\mathcal{A} = \begin{bmatrix} G_i(x_1) & P_i(x_1) & \dots & G_i(x_{m-1}) & P_i(x_{m-1}) \end{bmatrix}_{0 \leq i \leq 2m}.$$

Moreover, put

$$\Gamma_1^{a,b} := \left[\Gamma \left(r + s - \frac{1}{2} \right) \right]_{\substack{1 \leq r \leq m, r \neq a, \\ 1 \leq s \leq m, s \neq b}}, \quad \text{and} \quad \Gamma_2^{a,b} := \left[\Gamma \left(r + s + \frac{1}{2} \right) \right]_{\substack{0 \leq r \leq m-1, r \neq a, \\ 0 \leq s \leq m-1, s \neq b}};$$

(compare the definitions in Theorem 1.1).

- (1) Let $S \subset \{1, \dots, 2m\}$ be a subset with 2 elements and let \mathcal{A}^S be the $2(m-1) \times 2(m-1)$ -matrix that is obtained by removing from \mathcal{A} all the rows indexed by $S \cup \{0\}$. Then

$$\int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^S) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} = \begin{cases} (m-1)! 2^{m-1} \det(\Gamma_1^{a,b}), & \text{if } S = \{2a-1, 2b\} \text{ and } a \leq b \\ -(m-1)! 2^{m-1} \det(\Gamma_1^{a,b}), & \text{if } S = \{2a-1, 2b\} \text{ and } a > b \\ 0, & \text{if } S \text{ has any other form.} \end{cases}$$

- (2) Let $S \subset \{0, \dots, 2m-1\}$ be a subset with 2 elements and let \mathcal{A}^S be the $2(m-1) \times 2(m-1)$ -matrix that is obtained by removing from \mathcal{A} all the rows indexed by $S \cup \{2m\}$. Then

$$\int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^S) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} = \begin{cases} (m-1)! 2^{m-1} \det(\Gamma_2^{a,b}), & \text{if } S = \{2a, 2b+1\} \text{ and } a \leq b \\ -(m-1)! 2^{m-1} \det(\Gamma_2^{a,b}), & \text{if } S = \{2a, 2b+1\} \text{ and } a > b \\ 0, & \text{if } S \text{ has any other form.} \end{cases}$$

Proof. We first prove (1). Fix $S \subset \{1, \dots, 2m\}$. Then

$$(3.2) \quad \mathcal{A}^S = \begin{bmatrix} G_i(x_1) & P_i(x_1) & \dots & G_i(x_m) & P_i(x_{m-1}) \end{bmatrix}_{1 \leq i \leq 2m, i \notin S}.$$

Let us denote the quantity that we want to compute by Ξ :

$$(3.3) \quad \Xi := \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^S) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}.$$

To ease notation put

$$(3.4) \quad \mu := m - 1.$$

Furthermore, let us denote the elements in $\{1, \dots, 2m\} \setminus S$ in ascending order by $s_1 < \dots < s_{2\mu}$ and let $\Sigma_{2\mu}$ denote the group of permutations on $\{1, \dots, 2\mu\}$. Expanding the determinant of \mathcal{A}^S yields

$$(3.5) \quad \det(\mathcal{A}^S) = \sum_{\pi \in \Sigma_{2\mu}} \text{sgn}(\pi) \prod_{i=1}^{\mu} G_{s_{\pi(2i-1)}}(x_i) P_{s_{\pi(2i)}}(x_i).$$

Recall from (2.18) the definition of $\langle _, _ \rangle$. Plugging (3.5) into (3.3) and integrating over all the x_i we see that

$$(3.6) \quad \Xi = \sum_{\pi \in \Sigma_{2\mu}} \text{sgn}(\pi) \prod_{i=1}^{\mu} \langle G_{s_{\pi(2i-1)}}(x), P_{s_{\pi(2i)}}(x) \rangle.$$

From Lemma 2.2 we know that $\langle G_k(x), P_\ell(x) \rangle = 0$ whenever $k + \ell$ is even. This already proves that $\Xi = 0$, if S is not of the form $S = \{2a - 1, 2b\}$, because in this case we can't make a partition of $\{1, \dots, 2m\} \setminus S$ into pairs of numbers where one number is even and the other is odd. If, on the other hand, $S = \{2a - 1, 2b\}$ does contain one odd and two even elements, in (3.6) we may as well sum over the subset

$$\Sigma'_{2\mu} := \{ \pi \in \Sigma_{2\mu} \mid \forall i \in \{1, \dots, \mu\} : s_{\pi(2i-1)} + s_{\pi(2i)} \text{ is odd} \}.$$

Let $\mathcal{T} \subset \Sigma_{2\mu}$ be the subgroup generated by the set of transpositions $\{(1\ 2), (3\ 4), \dots, ((2\mu - 1)\ 2\mu)\}$. We define an equivalence relation on $\Sigma'_{2\mu}$ via:

$$\forall \pi, \sigma \in \Sigma'_{2\mu} : \pi \sim \sigma \Leftrightarrow \exists \tau \in \mathcal{T} : \pi = \sigma \tau.$$

Note that the multiplication with τ from the right is crucial here. Let $\mathcal{C} := \Sigma'_{2\mu} / \sim$ denote the set of equivalence classes of \mathcal{T} in $\Sigma'_{2\mu}$. A set of representatives for \mathcal{C} is

$$\mathcal{R} := \{ \pi \in \Sigma_{2\mu} \mid \forall i \in \{1, \dots, \mu\} : s_{\pi(2i-1)} \text{ is odd and } s_{\pi(2i)} \text{ is even} \}.$$

Making a partition of $\Sigma'_{2\mu}$ into the equivalence classes of \sim in (3.6) we get

$$\Xi = \sum_{\pi \in \mathcal{R}} \sum_{\tau \in \mathcal{T}} \text{sgn}(\pi \circ \tau) \prod_{i=1}^{\mu} \langle G_{s_{\pi \circ \tau(2i-1)}}(x), P_{s_{\pi \circ \tau(2i)}}(x) \rangle.$$

For a fixed $\pi \in \mathcal{R}$ and all $\tau \in \mathcal{T}$, by Lemma 2.3 (1) we have

$$\prod_{i=1}^{\mu} \langle G_{s_{\pi \circ \tau(2i-1)}}(x), P_{s_{\pi \circ \tau(2i)}}(x) \rangle = \text{sgn}(\tau) \prod_{i=1}^{\mu} \langle G_{s_{\pi(2i-1)}}(x), P_{s_{\pi(2i)}}(x) \rangle$$

so that

$$\Xi = 2^\mu \sum_{\pi \in \mathcal{R}} \text{sgn}(\pi) \prod_{i=1}^{\mu} \langle G_{s_{\pi(2i-1)}}(x), P_{s_{\pi(2i)}}(x) \rangle.$$

Let us investigate \mathcal{R} further. We denote the group of permutation on $\{1, \dots, \mu\}$ by Σ_μ . The group $\Sigma_\mu \times \Sigma_\mu$ acts transitively and faithful on \mathcal{R} via

$$\forall i : ((\sigma_1, \sigma_2) \cdot \pi)(2i - 1) := \pi(2\sigma_1(i) - 1) \quad \text{and} \quad ((\sigma_1, \sigma_2) \cdot \pi)(2i) := \pi(2\sigma_2(i))$$

This shows that that we have a bijection $\Sigma_\mu \times \Sigma_\mu \rightarrow \mathcal{R}$, $(\sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_2) \cdot \pi_\star$ where $\pi_\star \in \mathcal{R}$ is fixed. Moreover, for all $(\sigma_1, \sigma_2) \in \Sigma_\mu \times \Sigma_\mu$ we have $\text{sign}((\sigma_1, \sigma_2) \cdot \pi_\star) = \text{sgn}(\sigma_1)\text{sgn}(\sigma_2)\text{sign}(\pi_\star)$.

Let us denote $2k_i - 1 = s_{\pi_\star(2i-1)}$ and $2\ell_i = s_{\pi_\star(2i)}$. We choose π_\star uniquely by requiring $k_1 < k_2 < \dots < k_\mu$ and $\ell_1 < \ell_2 < \dots < \ell_\mu$. By doing so we get

$$\begin{aligned}
(3.7) \quad \Xi &= 2^\mu \text{sgn}(\pi_\star) \sum_{(\sigma_1, \sigma_2) \in \Sigma_\mu \times \Sigma_\mu} \text{sgn}(\sigma_1)\text{sgn}(\sigma_2) \prod_{i=1}^\mu \langle G_{2k_{\sigma_1(i)}-1}(x), P_{2\ell_{\sigma_2(i)}}(x) \rangle \\
&= 2^\mu \mu! \text{sgn}(\pi_\star) \sum_{\sigma \in \Sigma_\mu} \text{sgn}(\sigma) \prod_{i=1}^\mu \langle G_{2k_{\sigma(i)}-1}(x), P_{2\ell_i}(x) \rangle. \\
&= 2^\mu \mu! \text{sgn}(\pi_\star) \sum_{\sigma \in \Sigma_\mu} \text{sgn}(\sigma) \prod_{i=1}^\mu (-1)^{k_{\sigma(i)} + \ell_i} \Gamma(k_{\sigma(i)} + \ell_i - \tfrac{1}{2})
\end{aligned}$$

the last line by Lemma 2.3 (2). By construction we have $\bigcup_{i=1}^\mu \{2k_i - 1, 2\ell_i\} = \{1, \dots, 2m\} \setminus S$, so that

$$\{k_1, \dots, k_\mu\} = \{1, \dots, m\} \setminus \{a\}, \text{ and } \{\ell_1, \dots, \ell_\mu\} = \{1, \dots, m\} \setminus \{b\}.$$

Hence, for all $\sigma \in \Sigma_\mu$ we have

$$(3.8) \quad \prod_{i=1}^\mu (-1)^{k_{\sigma(i)} + \ell_i} = (-1)^{m(m+1) - a - b} = (-1)^{a+b}.$$

and, furthermore,

$$(3.9) \quad \text{sgn}(\pi_\star) = \begin{cases} (-1)^{a+b}, & \text{if } a \leq b \\ (-1)^{a+b-1}, & \text{if } a > b \end{cases}.$$

Moreover,

$$(3.10) \quad \sum_{\sigma \in \Sigma_\mu} \text{sgn}(\sigma) \prod_{i=1}^\mu \Gamma(k_{\sigma(i)} + \ell_i - \tfrac{1}{2}) = \det \left([\Gamma(k + \ell - \tfrac{1}{2})]_{\substack{1 \leq k \leq m, k \neq a, \\ 1 \leq \ell \leq m, \ell \neq b}} \right) = \det(\Gamma_1^{a,b}),$$

Putting together (3.7), (3.9), (3.8) and (3.10) proves (1).

Remark. Because we use the symbols k, ℓ frequently we decided to use $\Gamma_1^{a,b} = [\Gamma(r + s - \tfrac{1}{2})]_{\substack{1 \leq r \leq m, r \neq a, \\ 1 \leq s \leq m, s \neq b}}$ as notation.

We now prove (2). Fix $S \subset \{0, \dots, 2m-1\}$. Similar to (3.2) we have

$$\mathcal{A}^S = \begin{bmatrix} G_i(x_1) & P_i(x_1) & \dots & G_i(x_m) & P_i(x_m) \end{bmatrix}_{0 \leq i \leq 2m-1, i \notin S}.$$

Put $\tilde{G}_i(x) := G_{i-1}(x)$ and $\tilde{P}_i(x) := P_{i-1}(x)$, so that

$$\mathcal{A}^S = \begin{bmatrix} \tilde{G}_i(x_1) & \tilde{P}_i(x_1) & \dots & \tilde{G}_i(x_m) & \tilde{P}_i(x_m) \end{bmatrix}_{1 \leq i \leq 2m, i \notin \tilde{S}},$$

where $\tilde{S} \subset \{1, \dots, 2m\}$ is the set that is obtained from S by adding 1 to both elements of S . Observe that,

$$\langle \tilde{G}_k(x), \tilde{P}_\ell(x) \rangle = \langle G_{k-1}(x), P_{\ell-1} \rangle = -\langle G_{\ell-1}(x), P_{k-1} \rangle,$$

by Lemma 2.3 (1) and hence, by Lemma 2.3 (2),

$$\begin{aligned}
\langle \tilde{G}_k(x), \tilde{P}_\ell(x) \rangle &= \begin{cases} (-1)^{i+j+1} \Gamma(i + j - \tfrac{1}{2}), & \text{if } k = 2j + 1, \ell = 2i \\ 0, & \text{if } k + \ell \text{ is even} \end{cases} \\
&= \begin{cases} (-1)^{i+j} \Gamma(i + j - \tfrac{3}{2}), & \text{if } k = 2j - 1, \ell = 2i \\ 0, & \text{if } k + \ell \text{ is even} \end{cases}
\end{aligned}$$

We may now proceed as in the proof of (1) until (3.7), and conclude that

$$\int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^S) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} = \begin{cases} (m-1)! 2^{m-1} \det(\tilde{\Gamma}_2^{a', b'}), & \text{if } \tilde{S} = \{2a' - 1, 2b'\} \text{ and } a \leq b \\ -(m-1)! 2^{m-1} \det(\tilde{\Gamma}_2^{a', b'}), & \text{if } \tilde{S} = \{2a' - 1, 2b'\} \text{ and } a > b, \\ 0, & \text{if } \tilde{R} \text{ has any other form.} \end{cases}$$

where

$$\tilde{\Gamma}_2^{a', b'} := [\Gamma(k + \ell - \frac{3}{2})]_{\substack{1 \leq k \leq m, k \neq a' \\ 1 \leq \ell \leq m, \ell \neq b'}}$$

Note that

$$\tilde{\Gamma}_2^{a', b'} = [\Gamma(k + \ell + \frac{1}{2})]_{\substack{0 \leq k \leq m-1, k \neq a'-1 \\ 0 \leq \ell \leq m-1, \ell \neq b'-1}} = \Gamma_2^{a'-1, b'-1}.$$

If $\tilde{S} = \{2a' - 1, 2b'\}$ then, by definition, $S = \{2a, 2b + 1\}$, where $a = a' - 1$ and $b = b' - 1$. Hence,

$$\begin{aligned} & \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^S) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\ &= \begin{cases} (m-1)! 2^{m-1} \det(\Gamma_2^{a, b}), & \text{if } R = \{2a, 2b + 1\} \text{ and } a \leq b \\ -(m-1)! 2^{m-1} \det(\Gamma_2^{a, b}), & \text{if } R = \{2a, 2b + 1\} \text{ and } a > b, \\ 0, & \text{if } R \text{ has any other form.} \end{cases} \end{aligned}$$

This finishes the proof. \square

Proposition 3.3 and Proposition 3.4 below become important in Section 4.1 and Section 4.2, respectively.

Proposition 3.3. Recall from Section 2.3 the definition of $P_k(x)$ and $G_k(x)$. Let \mathcal{M} denote the matrix

$$\mathcal{M} := \begin{bmatrix} P_i(u) & [G_i(x_j) \ P_i(x_j)]_{j=1, \dots, m-1} & G_i(u) & G_i(\infty) \end{bmatrix}_{i=0, \dots, 2m}$$

We have

$$\begin{aligned} & \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\ &= \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i, j}) \det \begin{bmatrix} P_{2j}(u) & P_{2i-1}(u) \\ P_{2j-1}(u) & P_{2i-2}(u) \end{bmatrix}. \end{aligned}$$

where $\Gamma^{i, j} := [\Gamma(r + s - \frac{1}{2})]_{\substack{1 \leq s \leq m, s \neq j \\ 1 \leq r \leq m, r \neq i}}$.

Proof. Let us denote the quantity that we want to compute by Θ :

$$\Theta := \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}.$$

A permutation with negative sign of the columns of \mathcal{M} yields,

$$(3.11) \quad \det(\mathcal{M}) = -\det \begin{bmatrix} G_i(\infty) & G_i(u) & P_i(u) & [G_i(x_j) \ P_i(x_j)]_{j=1, \dots, m-1} \end{bmatrix}_{i=0, \dots, 2m}$$

By Lemma 2.2 we have $G_i(\infty) = 0$ for $i \geq 1$ and $G_0(\infty) = \sqrt{2\pi}$. Expanding the determinant in (3.11) with Laplace expansion we get

$$\det(\mathcal{M}) = -\sqrt{2\pi} \sum_{1 \leq k < \ell \leq 2m} (-1)^{k+\ell-1} (G_k(u) P_\ell(u) - G_\ell(u) P_k(u)) \det(\mathcal{A}^{k, \ell}),$$

where $\mathcal{A}^{k,\ell} := \begin{bmatrix} G_i(x_j) & P_i(x_j) \end{bmatrix}_{\substack{1 \leq i \leq 2m, i \notin \{k,\ell\} \\ 1 \leq j \leq m-1}}$. Hence, Θ is equal to

$$(3.12) \quad \sqrt{2\pi} \sum_{1 \leq k < \ell \leq 2m} (-1)^{k+\ell} (G_k(u)P_\ell(u) - G_\ell(u)P_k(u)) \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^{k,\ell}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}$$

In the notation of Lemma 3.2 we have $\mathcal{A}^{k,\ell} = \mathcal{A}^{\{k,\ell\}}$. Applying the Lemma 3.2 yields

$$\int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^{k,\ell}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} = \begin{cases} (m-1)! 2^{m-1} \det(\Gamma_1^{i,j}), & \text{if } \{k, \ell\} = \{2i-1, 2j\}, i \leq j. \\ -(m-1)! 2^{m-1} \det(\Gamma_1^{i,j}), & \text{if } \{k, \ell\} = \{2i-1, 2j\}, i > j. \\ 0, & \text{else.} \end{cases}$$

where $\Gamma_1^{i,j} = [\Gamma(r+s-\frac{1}{2})]_{\substack{1 \leq r \leq m, r \neq i \\ 1 \leq s \leq m, s \neq j}}$. When we want to plug this into (3.12) we have to incorporate that

$$\begin{cases} \text{If } k = 2i-1, \ell = 2j \text{ and } k < \ell, \text{ then } i \leq j. \\ \text{If } k = 2j, \ell = 2i-1 \text{ and } k < \ell, \text{ then } i > j. \end{cases}$$

From this we get

$$\begin{aligned} \Theta = & (-1) \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \left[\sum_{1 \leq i \leq j \leq m} \det(\Gamma^{i,j}) (G_{2i-1}(u)P_{2j}(u) - G_{2j}(u)P_{2i-1}(u)) \right. \\ & \left. - \sum_{1 \leq j < i \leq m} \det(\Gamma^{i,j}) (G_{2j}(u)P_{2i-1}(u) - G_{2i-1}(u)P_{2j}(u)) \right] \end{aligned}$$

By Lemma 2.2 we have $G_k(u) = -e^{-\frac{u^2}{2}} P_{k-1}(u)$, $k \geq 1$, which we can plug in into the upper expression to obtain

$$\begin{aligned} \Theta = & \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \left[\sum_{1 \leq i \leq j \leq m} \det(\Gamma^{i,j}) (P_{2i-2}(u)P_{2j}(u) - P_{2j-1}(u)P_{2i-1}(u)) \right. \\ & \left. - \sum_{1 \leq j < i \leq m} \det(\Gamma^{i,j}) (P_{2j-1}(u)P_{2i-1}(u) - P_{2i-2}(u)P_{2j}(u)) \right] \\ = & \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) (P_{2i-2}(u)P_{2j}(u) - P_{2j-1}(u)P_{2i-1}(u)) \\ = & \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \det \begin{bmatrix} P_{2j}(u) & P_{2i-1}(u) \\ P_{2j-1}(u) & P_{2i-2}(u) \end{bmatrix} \end{aligned}$$

This finishes the proof. \square

Proposition 3.4. Let \mathcal{M} denote the matrix $\mathcal{M} = \begin{bmatrix} P_i(u) & \begin{bmatrix} G_i(x_j) & P_i(x_j) \end{bmatrix}_{j=1, \dots, m-1} & G_i(u) \end{bmatrix}_{i=0, \dots, 2m-1}$. Then

$$\begin{aligned} & \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\ = & (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix}, \end{aligned}$$

where $\Gamma_2^{i,j} = [\Gamma(r+s-\frac{3}{2})]_{\substack{0 \leq r \leq m-1, r \neq i \\ 0 \leq s \leq m-1, s \neq j}}$.

Proof. The proof works similar as the the proof for Proposition 3.3: Again, we denote by Θ the quantity that we want to compute:

$$\Theta := \int_{x_1, \dots, x_m \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m.$$

We have

$$\det(\mathcal{M}) = -\det \begin{bmatrix} G_i(u) & P_i(u) & [G_i(x_j) & P_i(x_j)]_{j=1,\dots,m} \end{bmatrix}_{i=0,\dots,2m-1}.$$

Expanding the determinant with Laplace expansion we get

$$\det(\mathcal{M}) = - \sum_{0 \leq k < \ell \leq 2m-1} (-1)^{k+\ell-1} (G_k(u)P_\ell(u) - G_\ell(u)P_k(u)) \det(\mathcal{A}^{k,\ell}),$$

where

$$\mathcal{A}^{k,\ell} = [G_i(x_1) \ P_i(x_1) \ \dots \ G_i(x_m) \ P_i(x_m)]_{i=0,\dots,2m-1, i \notin \{k,\ell\}}$$

Hence,

$$(3.13) \quad \Theta = \sum_{0 \leq k < \ell \leq 2m-1} (-1)^{k+\ell} (G_k(u)P_\ell(u) - G_\ell(u)P_k(u)) \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^{k,\ell}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}.$$

By Lemma 3.2 (2) we have

$$\begin{aligned} & \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{A}^{k,\ell}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\ &= \begin{cases} (m-1)! 2^{m-1} \det(\Gamma_2^{i,j}), & \text{if } \{k, \ell\} = \{2i, 2j+1\} \text{ and } i \leq j \\ -(m-1)! 2^{m-1} \det(\Gamma_2^{i,j}), & \text{if } \{k, \ell\} = \{2i, 2j+1\} \text{ and } i > j \\ 0, & \text{else.} \end{cases} \end{aligned}$$

where

$$\Gamma_2^{i,j} = [\Gamma(r+s+\frac{1}{2})]_{\substack{0 \leq r \leq m-1, r \neq i \\ 0 \leq s \leq m-1, s \neq j}}.$$

When plugging this into (3.13) we must take into account that

$$\begin{cases} \text{If } k = 2i, \ell = 2j+1 \text{ and } k < \ell, \text{ then } i \leq j. \\ \text{If } k = 2j+1, \ell = 2i \text{ and } k < \ell, \text{ then } i > j. \end{cases}$$

This yields

$$\begin{aligned} \Theta &= (m-1)! 2^{m-1} \left[\sum_{0 \leq j < i \leq m-1} \det(\Gamma_2^{i,j}) (G_{2j+1}(u)P_{2i}(u) - G_{2i}(u)P_{2j+1}(u)) \right. \\ &\quad \left. - \sum_{0 \leq i < j \leq m-1} \det(\Gamma_2^{i,j}) (G_{2i}(u)P_{2j+1}(u) - G_{2j+1}(u)P_{2i}(u)) \right] \\ &= m! 2^{m-1} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) (P_{2i}(u)G_{2j+1}(u) - P_{2j+1}(u)G_{2i}(u)) \end{aligned}$$

Using from Lemma 2.2 that $G_k(u) = -e^{-\frac{u^2}{2}} P_{k-1}(u)$, we finally obtain

$$\Theta = (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2j+1}(u) & P_{2i}(u) \\ P_{2j}(u) & P_{2i-1}(u) \end{bmatrix}.$$

This finishes the proof. \square

4. PROOF OF THEOREM 1.1

All of the following section is inspired by the computations made in [15, Sec. 22]. Recall from (1.5) that we have put

$$\mathcal{I}_n(u) = \mathbb{E}_{A \sim \text{GOE}(n; u, 1)} |\det(A)|, \quad \text{and} \quad \mathcal{J}_n(u) = \mathbb{E}_{A \sim \text{GOE}(n; u, 1)} \det(A)$$

The proof of Theorem 1.1 is based on the idea to decompose $\mathcal{I}_n(u) = (\mathcal{I}_n(u) + \mathcal{J}_n(u)) - \mathcal{J}_n(u)$ and then to compute the two summands $\mathcal{I}_n(u) + \mathcal{J}_n(u)$ and $\mathcal{J}_n(u)$ separately. By definition of the Gaussian Orthogonal

Ensemble and since $\mathcal{I}_n(u) = \mathbb{E}_{A \sim \text{GOE}(n)} |\det(A - uI_n)|$ we have

$$\mathcal{I}_n(u) = \frac{1}{\sqrt{2}^n \sqrt{\pi}^{n(n+1)/2}} \int_{\substack{A \in \mathbb{R}^{n \times n} \\ \text{symmetric}}} |\det(A - uI_n)| e^{-\frac{1}{2} \text{Trace}(A^2)} dA.$$

By [16, Theorem 3.2.17], the density of the (ordered) eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$ of $A \sim \text{GOE}(n)$ is given by

$$\frac{\sqrt{\pi}^{\frac{n(n+1)}{2}}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \mathbf{1}_{\{\lambda_1 \leq \dots \leq \lambda_n\}},$$

where $\Delta(\lambda) := \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$ and $\mathbf{1}_{\{\lambda_1 \leq \dots \leq \lambda_n\}}$ is the characteristic function of the set $\{\lambda_1 \leq \dots \leq \lambda_n\}$. This implies

$$(4.1) \quad \mathcal{I}_n(u) = \frac{1}{\sqrt{2}^n \prod_{i=1}^n \Gamma(\frac{i}{2})} \int_{\lambda_1 \leq \dots \leq \lambda_n} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n |\lambda_i - u| d\lambda_1 \dots d\lambda_n.$$

Similiarly,

$$(4.2) \quad \mathcal{J}_n(u) = \frac{1}{\sqrt{2}^n \prod_{i=1}^n \Gamma(\frac{i}{2})} \int_{\lambda_1 \leq \dots \leq \lambda_n} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n (\lambda_i - u) d\lambda_1 \dots d\lambda_n.$$

For even n the integral in (4.2) can be expressed nicely in terms of the Hermite polynomials $H_k(x)$ from (2.8). The following is [15, Eq. (22.2.38)].

Theorem 4.1. *Let $n = 2m$ and $H_k(x)$ denote the Hermite polynomial from (2.8). We have*

$$\int_{\lambda_1 \leq \dots \leq \lambda_n} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n (\lambda_i - u) d\lambda_1 \dots d\lambda_n = \frac{\sqrt{\pi}^m}{2^{m^2}} \left[\prod_{i=1}^{m-1} (2i)! \right] H_{2m}(u),$$

Theorem 4.1 will be of use later when we compute $E(n, p)$ for even n . For odd n a similar but more involved formula can found in [15, Eq. (22.2.39)]. However, we decided not to put it here, because we do not need it for further computations.

In the remainder of the section we put.

$$(4.3) \quad C := \left(\sqrt{2}^n \prod_{i=1}^n \Gamma(\frac{i}{2}) \right)^{-1}$$

and $\lambda_0 := -\infty$. We can write (4.1) as

$$\mathcal{I}_n(u) = C \sum_{j=0}^n (-1)^j \int_{\substack{\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq u \\ u \leq \lambda_{j+1} \leq \dots \leq \lambda_n}} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n (\lambda_i - u) d\lambda_1 \dots d\lambda_n$$

and (4.2) as

$$\mathcal{J}_n(u) = C \sum_{j=0}^n \int_{\substack{\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq u \\ u \leq \lambda_{j+1} \leq \dots \leq \lambda_n}} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n (\lambda_i - u) d\lambda_1 \dots d\lambda_n.$$

Hence,

$$(4.4) \quad \mathcal{I}_n(u) + \mathcal{J}_n(u) = 2C \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \int_{\substack{\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{2j} \leq u \\ u \leq \lambda_{2j+1} \leq \dots \leq \lambda_n}} \Delta(\lambda) e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} \prod_{i=1}^n (\lambda_i - u) d\lambda_1 \dots d\lambda_n$$

We write $\Delta(\lambda) \prod_{i=1}^n (\lambda_i - u)$ as a Vandermonde determinant:

$$\Delta(\lambda) \prod_{i=1}^n (\lambda_i - u) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^n (\lambda_i - u) = \det \begin{bmatrix} u^k & \lambda_1^k & \dots & \lambda_n^k \end{bmatrix}_{k=0, \dots, n}.$$

Since we may add arbitrary multiple of rows to other rows of a matrix without changing its determinant, we have

$$(4.5) \quad \Delta(\lambda) \prod_{i=1}^n (\lambda_i - u) = \det \begin{bmatrix} P_k(u) & P_k(\lambda_1) & \dots & P_k(\lambda_n) \end{bmatrix}_{k=0, \dots, n},$$

where the $P_k(x)$, $k = 0, 1, \dots, n$, are the Hermite polynomials from (2.16). Plugging this into (4.4) yields

$$(4.6) \quad \mathcal{I}_n(u) + \mathcal{J}_n(u) = 2C \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \int_{\substack{\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{2j} \leq u \\ u \leq \lambda_{2j+1} \leq \dots \leq \lambda_n}} \det \begin{bmatrix} P_k(u) & P_k(\lambda_1) & \dots & P_k(\lambda_n) \end{bmatrix}_{k=0, \dots, n} e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} d\lambda_1 \dots d\lambda_n$$

We now distinguish the cases n even and n odd.

4.1. The case when n is even. Recall that we have put $n = 2m$, so that $\lfloor \frac{n}{2} \rfloor = m$. Moreover, recall from (2.17) that we have put

$$G_k(x) = \int_{-\infty}^x P_k(y) e^{-\frac{y^2}{2}} dy$$

Observe that each λ_i appears in exactly one column on the right hand side of (4.5). Integrating over $\lambda_1, \lambda_3, \lambda_5, \dots$ in (4.6) therefore yields

$$(4.7) \quad \mathcal{I}_n(u) + \mathcal{J}_n(u) = 2C \sum_{j=0}^m \int_{\substack{\lambda_2 \leq \lambda_4 \leq \dots \leq \lambda_{2j} \leq u \\ u \leq \lambda_{2j+2} \leq \dots \leq \lambda_{2m}}} \det(\mathcal{N}_j) e^{-\sum_{i=1}^m \frac{\lambda_{2i}^2}{2}} d\lambda_2 \dots d\lambda_{2m}$$

where \mathcal{N}_j is the matrix

$$\begin{aligned} \mathcal{N}_j = & \begin{bmatrix} P_k(u) & [G_k(\lambda_{2i}) - G_k(\lambda_{2i-2}) & P_k(\lambda_{2i})]_{i=1, \dots, j} & [G_k(\lambda_{2j+2}) - G_k(u) & P_k(\lambda_{2j+2})] & \dots \\ \dots & [G_k(\lambda_{2i}) - G_k(\lambda_{2i-2}) & P_k(\lambda_{2i})]_{i=j+2, \dots, m} \end{bmatrix}_{k=0, \dots, n} \end{aligned}$$

Adding the first column of \mathcal{N}_j to its third column, and the result to the fifth column and so on, does not change the value of the determinant. Hence, $\det(\mathcal{N}_j) = \det(\mathcal{M}_j)$, where

$$(4.8) \quad \mathcal{M}_j := \begin{bmatrix} P_k(u) & [G_k(\lambda_{2i}) & P_k(\lambda_{2i})]_{i=1, \dots, j} & [G_k(\lambda_{2i}) - G_k(u) & P_k(\lambda_{2i})]_{i=j+1, \dots, m} \end{bmatrix}_{k=0, \dots, n}$$

Observe that each λ_{2i} appears in exactly two columns of \mathcal{M}_j . Hence, making a change of variables by interchanging λ_{2i} and $\lambda_{2i'}$ for any two i, i' does not change the value of the determinant of \mathcal{M}_j . Writing $x_i := \lambda_{2i}$, for $1 \leq i \leq m$, we therefore have

$$(4.9) \quad \mathcal{I}_m(u) + \mathcal{J}_m(u) = \frac{2C}{m!} \sum_{j=0}^m \binom{m}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} \det(\mathcal{M}_j) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m,$$

Using the multilinearity of the determinant we can write $\det(\mathcal{M}_j)$ as a sum of determinants of matrices, each of which double columns either equal $[G_k(x_i) \ P_k(x_i)]$ or $[-G_k(u) \ P_k(x_i)]$. Observe that whenever the column $[-G_k(u) \ P_k(x_i)]$ appears twice in a matrix the corresponding determinant equals zero. Moreover, we may interchange the double columns as we wish without changing the value of the determinant. All this yields

$$(4.10) \quad \det(\mathcal{M}_j) = \det(\mathcal{K}) - (m - j) \det(\mathcal{L}),$$

where

$$\begin{aligned} \mathcal{K} &= \begin{bmatrix} P_k(u) & [G_k(x_i) & P_k(x_i)]_{i=1, \dots, m} \end{bmatrix}_{k=0, \dots, 2m}, \\ \mathcal{L} &= \begin{bmatrix} P_k(u) & [G_k(x_i) & P_k(x_i)]_{i=1, \dots, m-1} & G_k(u) & P_k(x_m) \end{bmatrix}_{k=0, \dots, 2m}. \end{aligned}$$

Note that $\mathcal{K} e^{-\sum_{i=1}^m x_i^2}$ is invariant under any permutations of the x_i . We may apply Lemma 3.1 to conclude that

$$\begin{aligned}
(4.11) \quad & \frac{2C}{m!} \sum_{j=0}^m \binom{m}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} \det(\mathcal{K}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m \\
&= \frac{2C}{m!} \int_{x_1, \dots, x_m \in \mathbb{R}} \det(\mathcal{K}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m \\
&= 2C \int_{x_1 < \dots < x_m \in \mathbb{R}} \det(\mathcal{K}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m \\
&= 2C \int_{\lambda_1 < \dots < \lambda_n \in \mathbb{R}} \det \begin{bmatrix} P_k(u) & P_k(\lambda_1) & \dots & P_k(\lambda_n) \end{bmatrix}_{k=0, \dots, n} e^{-\sum_{i=1}^m \frac{\lambda_i^2}{2}} d\lambda_1 \dots d\lambda_n \\
&= 2C \int_{\lambda_1 < \dots < \lambda_n \in \mathbb{R}} \left[\prod_{i=1}^n (\lambda_i - u) \right] \Delta(\lambda) e^{-\sum_{i=1}^m \frac{\lambda_i^2}{2}} d\lambda_1 \dots d\lambda_n \\
&= 2\mathcal{J}_n(u).
\end{aligned}$$

the fifth line by (4.5) and the last line by (4.2). Combining this with (4.9) and (4.10) we see that

$$\begin{aligned}
\mathcal{I}_n(u) &= (\mathcal{I}_n(u) + \mathcal{J}_n(u)) - \mathcal{J}(u) \\
&= \mathcal{J}_n(u) - \frac{2C}{m!} \sum_{j=0}^m \binom{m}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} (m-j) \det(\mathcal{L}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m \\
&= \mathcal{J}_n(u) - \frac{2C}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_m}} \det(\mathcal{L}) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_m
\end{aligned}$$

Since $\det(\mathcal{L}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}}$ is invariant under permuting x_1, \dots, x_{m-1} (excluding x_m !) we may apply Lemma 3.1 to obtain

$$(4.12) \quad \mathcal{I}_n(u) = \mathcal{J}_n(u) - \frac{2C}{(m-1)!} \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \left[\int_{x_m=u}^{\infty} \det(\mathcal{L}) e^{-\frac{x_m^2}{2}} dx_m \right] e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}.$$

Observe that x_m appears in one single column in \mathcal{L} . Integrating over x_m in therefore shows that

$$\begin{aligned}
& \int_{x_m=u}^{\infty} \det(\mathcal{L}) e^{-\frac{x_m^2}{2}} dx_m \\
&= \det \begin{bmatrix} P_k(u) & [G_k(x_i) P_k(x_i)]_{i=1, \dots, m-1} & G_k(u) & G_k(\infty) - G_k(u) \end{bmatrix}_{k=0, \dots, 2m} \\
&= \det \underbrace{\begin{bmatrix} P_k(u) & [G_k(x_i) P_k(x_i)]_{i=1, \dots, m-1} & G_k(u) & G_k(\infty) \end{bmatrix}_{k=0, \dots, 2m}}_{=: \mathcal{M}}.
\end{aligned}$$

From Proposition 3.3 we get

$$\begin{aligned}
& \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\
&= \sqrt{2\pi} (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \det \begin{bmatrix} P_{2j}(u) & P_{2i-1}(u) \\ P_{2j-1}(u) & P_{2i-2}(u) \end{bmatrix}.
\end{aligned}$$

where $\Gamma^{i,j} := [\Gamma(r+s-\frac{1}{2})]_{\substack{1 \leq s \leq m, s \neq j \\ 1 \leq r \leq m, r \neq i}}.$

Hence, by (4.12):

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) - C \sqrt{2\pi} 2^m e^{-\frac{u^2}{2}} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \det \begin{bmatrix} P_{2j}(u) & P_{2i-1}(u) \\ P_{2j-1}(u) & P_{2i-2}(u) \end{bmatrix}$$

Finally, we substitute $C = \left(\sqrt{2}^n \prod_{i=1}^n \Gamma\left(\frac{i}{2}\right) \right)^{-1}$ (see (4.3)) and put the minus into the determinant to obtain

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) + \frac{\sqrt{2\pi} e^{-\frac{u^2}{2}}}{\prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix}$$

This finishes the proof.

4.2. The case when n is odd. Here we have $n = 2m - 1$ and hence $\lfloor \frac{n}{2} \rfloor = m - 1$. We proceed as in the preceding section and can therefore be brief in our explanations. In (4.6) we integrate over all the λ_i with i odd to obtain

$$(4.13) \quad \mathcal{I}_n(u) + \mathcal{J}_n(u) = 2C \sum_{j=0}^{m-1} \int_{\substack{x_1 \leq x_2 \leq \dots \leq x_j \leq u \\ u \leq x_{j+1} \leq \dots \leq x_{m-1}}} \det(\mathcal{N}_j) e^{-\sum_{i=1}^m \frac{x_i^2}{2}} dx_1 \dots dx_{m-1},$$

where $x_i := y_{2i}$, $1 \leq i \leq m - 1$ and \mathcal{N}_j is the matrix

$$\mathcal{N}_j = \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) - G_k(x_{i-1}) & P_k(x_i) \end{bmatrix}_{i=1, \dots, j} & \begin{bmatrix} G_k(x_{j+1}) - G_k(u) & P_k(x_{j+1}) \end{bmatrix} & \dots \\ \dots & \begin{bmatrix} G_k(x_i) - G_k(x_{i-1}) & P_k(x_i) \end{bmatrix}_{i=j+2, \dots, m-1} & G_k(\infty) - G_k(x_{m-1}) \end{bmatrix}_{k=0, \dots, n}.$$

We have $\det(\mathcal{N}_j) = \det(\mathcal{M}_j)$, where

$$\mathcal{M}_j = \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) & P_k(x_i) \end{bmatrix}_{i=1, \dots, j} & \begin{bmatrix} G_k(x_i) - G_k(u) & P_k(x_i) \end{bmatrix}_{i=j+1, \dots, m-1} & G_k(\infty) - G_k(u) \end{bmatrix}_{k=0, \dots, n}$$

Permuting x_1, \dots, x_j or permuting x_{j+1}, \dots, x_m does not change the value of $\det(\mathcal{M}_j)$, so that

$$(4.14) \quad \mathcal{I}_n(u) + \mathcal{J}_n(u) = \frac{2C}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-1}}} \det(\mathcal{M}_j) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1},$$

Using the multilinearity of the determinant we have

$$\det(\mathcal{M}_j) = \det(\mathcal{K}) - \det(\mathcal{M}) - (m-1-j) \det(\mathcal{L}),$$

where

$$\begin{aligned} \mathcal{K} &= \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) & P_k(x_i) \end{bmatrix}_{i=1, \dots, m-1} & G_k(\infty) \end{bmatrix}_{k=0, \dots, 2m-1}, \\ \mathcal{M} &= \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) & P_k(x_i) \end{bmatrix}_{i=1, \dots, m-1} & G_k(u) \end{bmatrix}_{k=0, \dots, 2m-1}, \\ \mathcal{L} &= \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) & P_k(x_i) \end{bmatrix}_{i=1, \dots, m-2} & G_k(u) & P_k(x_{m-1}) & G_k(\infty) - G_k(u) \end{bmatrix}_{k=0, \dots, 2m-1}. \end{aligned}$$

Integrating $\int_{x_{m-1} > u} \det(\mathcal{L}) e^{-\frac{x_{m-1}^2}{2}} dx$ replaces the $P_k(x_{m-1})$ in \mathcal{L} by $G_k(\infty) - G_k(u)$. Hence,

$$\begin{aligned} & \int_{x_{m-1}=u}^{\infty} \det(\mathcal{L}) e^{-\frac{x_{m-1}^2}{2}} dx_{m-1} \\ &= \det \begin{bmatrix} P_k(u) & \begin{bmatrix} G_k(x_i) & P_k(x_i) \end{bmatrix}_{i=1, \dots, m-2} & G_k(u) & G_k(\infty) - G_k(u) & G_k(\infty) - G_k(u) \end{bmatrix}_{k=0, \dots, 2m-1} \\ &= 0 \end{aligned}$$

and thus

$$\begin{aligned}
& \frac{2C}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-1}}} (m-1-j) \det(\mathcal{L}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\
&= \frac{2C}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-2}}} \left[\int_{x_{m-1}=u}^{\infty} \det(\mathcal{L}) e^{-\frac{x_{m-1}^2}{2}} dx_{m-1} \right] e^{-\sum_{i=1}^{m-2} \frac{x_i^2}{2}} dx_1 \dots dx_{m-2} \\
&= 0.
\end{aligned}$$

Using this, (4.14) becomes

$$\mathcal{I}_n(u) + \mathcal{J}_n(u) = \frac{2C}{(m-1)!} \sum_{i=0}^{m-1} \binom{m-1}{j} \int_{\substack{x_1, \dots, x_j \leq u \\ u \leq x_{j+1}, \dots, x_{m-1}}} (\det(\mathcal{K}) + \det(\mathcal{M})) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}$$

By construction, both $\det(\mathcal{K})$ and $\det(\mathcal{M})$ are invariant under any permutation of the x_i . We may apply Lemma 3.1 to get

$$\mathcal{I}_n(u) + \mathcal{J}_n(u) = \frac{2C}{(m-1)!} \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} (\det(\mathcal{K}) - \det(\mathcal{M})) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}$$

Similar to (4.11) we deduce that

$$\frac{2C}{(m-1)!} \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{K}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} = 2\mathcal{J}_n(u),$$

so that

$$(4.15) \quad \mathcal{I}_n(u) = (\mathcal{I}_n(u) + \mathcal{J}_n(u)) - \mathcal{J}_n(u) = \mathcal{J}_n(u) - \frac{2C}{(m-1)!} \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1}$$

By Proposition 3.4 we have

$$\begin{aligned}
& \int_{x_1, \dots, x_{m-1} \in \mathbb{R}} \det(\mathcal{M}) e^{-\sum_{i=1}^{m-1} \frac{x_i^2}{2}} dx_1 \dots dx_{m-1} \\
&= (m-1)! 2^{m-1} e^{-\frac{u^2}{2}} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2j+1}(u) & P_{2i}(u) \\ P_{2j}(u) & P_{2i-1}(u) \end{bmatrix},
\end{aligned}$$

where $\Gamma_2^{i,j} = [\Gamma(r+s+\frac{1}{2})]_{\substack{0 \leq r \leq m-1, r \neq i \\ 0 \leq s \leq m-1, s \neq j}}$. Combining this with (4.15), substituting $C = \left(\sqrt{2}^n \prod_{i=1}^n \Gamma(\frac{i}{2}) \right)^{-1}$ (see (4.3)) and putting the minus into the determinant we get

$$\mathcal{I}_n(u) = \mathcal{J}_n(u) + \frac{\sqrt{2} e^{-\frac{u^2}{2}}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix}.$$

This finishes the proof.

5. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. Recall from (1.3) and (1.4) that

$$\begin{aligned}
(5.1) \quad E(n, p) &= \frac{\sqrt{\pi} \sqrt{p-1}^{n-1}}{\Gamma(\frac{n}{2})} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathbb{E}_{A \sim \text{GOE}(n-1; u, 1)} |\det(A)| \\
&= \frac{\sqrt{\pi} \sqrt{p-1}^{n-1}}{\Gamma(\frac{n}{2})} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{I}_{n-1}(u), \quad \text{where } \sigma^2 = \frac{p}{2(p-1)}.
\end{aligned}$$

We now have to distinguish between the cases n even and n odd. The distinction between those cases is due to the nature of Theorem 1.1: The formula for $\mathcal{I}_{n-1}(u)$ depends on the parity of n .

5.1. Proof of Theorem 1.2 (1). In this case we have $n = 2m + 1$ and hence $n - 1 = 2m$. We know from Theorem 1.1 (1) that

$$(5.2) \quad \mathcal{I}_{n-1}(u) = \mathcal{J}_{2m}(u) + \frac{\sqrt{2\pi} e^{-\frac{u^2}{2}}}{\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{2}\right)} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix}$$

Thus taking the expectation over $\mathcal{I}_{n-1}(u)$ we may take the expectation over the two summands above. Before we compute the expectation of $\mathcal{J}_{2m}(u)$ in Lemma 5.1 below, however, we first have to proof the following lemma.

Lemma 5.1. *For all $m \geq 1$ we have*

- (1) $\sqrt{\pi} (2(m-1))! (2m-1) = 2^{2m-1} \Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)$.
- (2) $\sqrt{\pi}^{m+1} \left[\prod_{i=1}^{m-1} (2i)! \right] (2m)! = m! 2^{m(m+1)} \prod_{i=1}^{2m+1} \Gamma\left(\frac{i}{2}\right)$.

Proof. Throughout the proof we will have to use the identities $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ [19, 43:4:3] and $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ [19, 43:4:3].

We prove both claims using an induction argument. For (1) and $m = 1$ we have

$$\frac{\sqrt{\pi} (2(m-1))! (2m-1)}{2^{2m-1} \Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

For $m > 1$, using the induction hypothesis, we have

$$\begin{aligned} \frac{\sqrt{\pi} (2(m-1))! (2m-1)}{2^{2m-1} \Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)} &= \frac{(2m-2)(2m-3)}{4} \frac{2m-1}{2m-3} \frac{\Gamma\left(\frac{2m-1}{2}\right) \Gamma(m-1)}{\Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)} \\ &= \frac{(2m-2)(2m-1)}{4} \frac{2}{2m-1} \frac{1}{m-1} = 1 \end{aligned}$$

For (2) and $m = 1$ we have

$$\sqrt{\pi}^{m+1} \left[\prod_{i=1}^{m-1} (2i)! \right] (2m)! = 2\pi = m! 2^{m(m+1)} \prod_{i=1}^{2m+1} \Gamma\left(\frac{i}{2}\right)$$

For $m > 1$, using the induction hypothesis, we have

$$\frac{\sqrt{\pi}^{m+1} \left[\prod_{i=1}^{m-1} (2i)! \right] (2m)!}{m! 2^{m(m+1)} \prod_{i=1}^{2m+1} \Gamma\left(\frac{i}{2}\right)} = \frac{\sqrt{\pi} (2(m-1))! 2m(2m-1)}{m 2^{2m} \Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)} = \frac{\sqrt{\pi} (2(m-1))! (2m-1)}{2^{2m-1} \Gamma\left(\frac{2m+1}{2}\right) \Gamma(m)} = 1,$$

the last equality because of (1). This finishes the proof. \square

Using Lemma 5.1 we can prove the following.

Lemma 5.2. *We have*

$$\frac{\sqrt{\pi} \sqrt{p-1}^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{J}_{2m}(u) = 1$$

Proof. A combination of Theorem 4.1 and (4.2) reveals that

$$\mathcal{J}_{2m}(u) = \frac{\sqrt{\pi}^m \left[\prod_{i=1}^{m-1} (2i)! \right]}{2^{m(m+1)} \prod_{i=1}^{2m} \Gamma\left(\frac{i}{2}\right)} H_{2m}(u).$$

By Lemma 2.4 (1) we have $\mathbb{E}_{u \sim N(0, \sigma^2)} H_{2m}(u) = \frac{(2m)!}{m!} (2\sigma^2 - 1)^m$. Plugging in $\sigma^2 = \frac{p}{2(p-1)}$ yields

$$\mathbb{E}_{u \sim N(0, \sigma^2)} H_{2m}(u) = \frac{(2m)!}{m! (p-1)^m}$$

Thus

$$\begin{aligned} \frac{\sqrt{\pi}\sqrt{p-1}^{n-1}}{\Gamma(\frac{n}{2})} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{J}_{2m}(u) &= \frac{\sqrt{\pi}\sqrt{p-1}^{n-1}}{\Gamma(\frac{n}{2})} \frac{\sqrt{\pi}^m \left[\prod_{i=1}^{m-1} (2i)! \right]}{2^{m(m+1)} \prod_{i=1}^{2m} \Gamma(\frac{i}{2})} \frac{(2m)!}{m!(p-1)^m} \\ &= \frac{\sqrt{\pi}^{m+1} \left[\prod_{i=1}^{m-1} (2i)! \right]}{2^{m(m+1)} \prod_{i=1}^n \Gamma(\frac{i}{2})} \frac{(2m)!}{m!} = 1 \end{aligned}$$

the last equality by Lemma 5.1 (2). \square

Lemma 5.2 in combination with (5.1) and (5.2) shows that $E(n, p)$ equals

$$(5.3) \quad 1 + \frac{\sqrt{2\pi}\sqrt{p-1}^{n-1}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \sum_{1 \leq i, j \leq m} \det(\Gamma^{i,j}) \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix}.$$

Applying Lemma 5.3 below to (5.3) we get finally get that $E(n, p)$ equals

$$1 + \frac{\sqrt{\pi}\sqrt{p-1}^{n-2}\sqrt{3p-2}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \sum_{1 \leq i, j \leq m} \frac{\det(\Gamma^{i,j}) \Gamma(i+j-\frac{1}{2})}{\frac{3-2i-2j}{1-2i+2j} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j-1}} F\left(2-2i, 1-2j, \frac{5}{2}-i-j, \frac{3p-2}{4(p-1)}\right),$$

which is the statement from Theorem 1.2 (1).

Lemma 5.3. For any $1 \leq i, j \leq m$ we have

$$\begin{aligned} &\mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix} \\ &= \frac{\det(\Gamma^{i,j}) \Gamma(i+j-\frac{1}{2})}{\sqrt{2\pi} \frac{3-2i-2j}{1-2i+2j} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j-1}} \sqrt{\frac{3p-2}{p-1}} F\left(2-2i, 1-2j, \frac{5}{2}-i-j, \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

Proof. Write

$$\mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix} = \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} (P_{2j-1}(u)P_{2i-1}(u) - P_{2i-2}(u)P_{2j}(u))$$

From Lemma 2.4 (2) we get for all $1 \leq i \leq m$ that

$$\begin{aligned} (5.4) \quad &\mathbb{E}_{u \sim N(0, \sigma^2)} P_{2i-1}(u)P_{2j-1}(u) e^{-\frac{u^2}{2}} \\ &= \frac{(-1)^{i+j-1} 2^{i+j-1} \Gamma(i+j-1+\frac{1}{2})}{\sqrt{\pi} (\sigma^2+1)^{i+j-1+\frac{1}{2}}} F\left(1-2i, 1-2j; \frac{1}{2}-i-j+1; \frac{3p-2}{4(p-1)}\right) \\ &= \frac{(-1)^{i+j-1} 4^{i+j} \Gamma(i+j-\frac{1}{2})}{2\sqrt{2\pi}} \left(\frac{p-1}{3p-2}\right)^{i+j-\frac{1}{2}} F\left(1-2i, 1-2j; \frac{3}{2}-i-j; \frac{3p-2}{4(p-1)}\right), \end{aligned}$$

and

$$\begin{aligned} (5.5) \quad &\mathbb{E}_{u \sim N(0, \sigma^2)} P_{2i}(u)P_{2j-2}(u) e^{-\frac{u^2}{2}} \\ &= \frac{(-1)^{i+j-1} 2^{i+j-1} \Gamma(i+j-1+\frac{1}{2})}{\sqrt{\pi} (\sigma^2+1)^{i+j-1+\frac{1}{2}}} F\left(-2i, 2-2j; \frac{1}{2}-i-j+1; \frac{3p-2}{4(p-1)}\right) \\ &= \frac{(-1)^{i+j-1} 4^{i+j} \Gamma(i+j-\frac{1}{2})}{2\sqrt{2\pi}} \left(\frac{p-1}{3p-2}\right)^{i+j-\frac{1}{2}} F\left(-2i, 2-2j; \frac{3}{2}-i-j; \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

Thus

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} (P_{2j-1}(u)P_{2i-1}(u) - P_{2i-2}(u)P_{2j}(u)) \\ &= \det(\Gamma^{i,j}) \frac{(-1)^{i+j-1} 4^{i+j} \Gamma(i+j-\frac{1}{2})}{2\sqrt{2\pi}} \left(\frac{p-1}{3p-2} \right)^{i+j-\frac{1}{2}} \\ & \quad \left[F\left(1-2i, 1-2j; \frac{3}{2}-i-j; \frac{3p-2}{4(p-1)}\right) - F\left(2-2i, -2j; \frac{3}{2}-i-j; \frac{3p-2}{4(p-1)}\right) \right]. \end{aligned}$$

By Lemma 2.1 we have

$$\begin{aligned} & F\left(1-2i, 1-2j; \frac{3}{2}-i-j; x\right) - F\left(2-2i, -2j; \frac{3}{2}-i-j; x\right) \\ &= 2x \frac{1-2i+2j}{3-2i-2j} F\left(2-2i, 1-2j, \frac{5}{2}-i-j, x\right). \end{aligned}$$

This shows that

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i-1}(u) & P_{2j}(u) \\ P_{2i-2}(u) & P_{2j-1}(u) \end{bmatrix} \\ &= \frac{\det(\Gamma^{i,j}) \Gamma(i+j-\frac{1}{2})}{\sqrt{2\pi} \frac{3-2i-2j}{1-2i+2j} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j-1}} \sqrt{\frac{3p-2}{p-1}} F\left(2-2i, 1-2j, \frac{5}{2}-i-j, \frac{3p-2}{4(p-1)}\right), \end{aligned}$$

which finishes the proof. \square

5.2. The case $n = 2m$ is even. In this case we have $n-1 = 2m-1$, so that (5.1) becomes

$$E(n, p) = \frac{\sqrt{\pi} \sqrt{p-1}^{n-1}}{\Gamma(n)} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{I}_{2m-1}(u), \quad \text{where } \sigma^2 = \frac{p}{2(p-1)}.$$

We apply Theorem 1.1 (2) to obtain

$$\mathcal{I}_{2m-1}(u) = \mathcal{J}_{2m-1}(u) + \frac{\sqrt{2} e^{-\frac{u^2}{2}}}{\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{2}\right)} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix}$$

where $\Gamma_2^{i,j} = [\Gamma(r+s+\frac{1}{2})]_{\substack{0 \leq r \leq m-1, r \neq i \\ 0 \leq s \leq m-1, s \neq j}}$. Since the normal distribution is symmetric around the origin we have

$$\begin{aligned} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{J}_{2m-1}(u) &= \mathbb{E}_{u \sim N(0, \sigma^2)} \mathbb{E}_{A \sim \text{GOE}(2m-1)} \det(A - uI_{2m-1}) \\ &= (-1)^{2m-1} \mathbb{E}_{u \sim N(0, \sigma^2)} \mathbb{E}_{A \sim \text{GOE}(2m-1)} \det((-A) - (-u)I_{2m-1}) \\ &= - \mathbb{E}_{u \sim N(0, \sigma^2)} \mathbb{E}_{A \sim \text{GOE}(2m-1)} \det(A - uI_{2m-1}) \\ &= - \mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{J}_{2m-1}(u), \end{aligned}$$

and hence $\mathbb{E}_{u \sim N(0, \sigma^2)} \mathcal{J}_{2m-1}(u) = 0$. This shows that

$$(5.6) \quad E(n, p) = \frac{\sqrt{2\pi} \sqrt{p-1}^{n-1}}{\prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)} \sum_{0 \leq i, j \leq m-1} \det(\Gamma_2^{i,j}) \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix}$$

Applying Lemma 5.4 below to (5.6) we see that $E(n, p)$ equals

$$\begin{aligned} & \frac{\sqrt{p-1}^{n-2} \sqrt{3p-2}}{\prod_{i=1}^n \Gamma\left(\frac{i}{2}\right)} \sum_{j=0}^{m-1} \left[\frac{\sqrt{\pi} \det(\Gamma_2^{0,j}) (2j+1)!}{(-1)^j 2^{2j} j!} \frac{(p-2)^j p}{(p-1)^j (3p-2)} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right) \right. \\ & \quad \left. - \frac{\det(\Gamma_2^{0,j}) \Gamma(j+\frac{1}{2})}{\left(-\frac{3p-2}{4(p-1)}\right)^{j+1}} + \sum_{i=1}^{m-1} \frac{\det(\Gamma_2^{i,j}) \Gamma(i+j+\frac{1}{2})}{\frac{(1-2i-2j)}{(1-2i+2j)} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j}} F\left(-2j, -2i+1, \frac{3}{2}-i-j, \frac{3p-2}{4(p-1)}\right) \right]. \end{aligned}$$

which proves Theorem 1.2 (2).

Lemma 5.4. For all $0 \leq i \leq m-1$ and $0 \leq j \leq m-1$ the following holds.

(1) If $i > 0$:

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix} \\ &= \frac{\Gamma(i+j+\frac{1}{2})}{\sqrt{2\pi} \frac{(1-2i-2j)}{(1-2i+2j)} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j}} \sqrt{\frac{3p-2}{p-1}} F\left(-2j, -2i+1, \frac{3}{2}-i-j, \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

(2) If $i = 0$:

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix} \\ &= \sqrt{\frac{3p-2}{p-1}} \left[\frac{(-1)^j (2j+1)!}{2^{2j} \sqrt{2} j!} \frac{(p-2)^j p}{(p-1)^j (3p-2)} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right) - \frac{\Gamma(j+\frac{1}{2})}{\sqrt{2\pi} \left(-\frac{3p-2}{4(p-1)}\right)^{j+1}} \right]. \end{aligned}$$

Proof. Write

$$\mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix} = \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} (P_{2i}(u)P_{2j}(u) - P_{2i-1}(u)P_{2j+1}(u))$$

We prove (1). Fix $0 < i \leq m$ and $0 \leq j \leq m$. By Lemma 2.5 (1) and similar to (5.5) we have

$$\begin{aligned} (5.7) \quad & \mathbb{E}_{u \sim N(0, \sigma^2)} P_{2i}(u)P_{2j}(u)e^{-\frac{u^2}{2}} \\ &= \frac{(-1)^{i+j} 4^{i+j+1} \Gamma(i+j+\frac{1}{2})}{2\sqrt{2\pi}} \left(\frac{p-1}{3p-2}\right)^{i+j+\frac{1}{2}} F\left(-2i, -2j; \frac{1}{2}-i-j; \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

and, similar to (5.4),

$$\begin{aligned} (5.8) \quad & \mathbb{E}_{u \sim N(0, \sigma^2)} P_{2i-1}(u)P_{2j+1}(u)e^{-\frac{u^2}{2}} \\ &= \frac{(-1)^{i+j} 4^{i+j+1} \Gamma(i+j+\frac{1}{2})}{2\sqrt{2\pi}} \left(\frac{p-1}{3p-2}\right)^{i+j+\frac{1}{2}} F\left(-(2i-1), -(2j+1); \frac{1}{2}-i-j; \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

This shows that

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} (P_{2i}(u)P_{2j}(u) - P_{2i-1}(u)P_{2j+1}(u)) \\ &= \frac{4}{2\sqrt{2\pi}} \Gamma\left(i+j+\frac{1}{2}\right) \left(-\frac{4(p-1)}{3p-2}\right)^{i+j} \sqrt{\frac{p-1}{3p-2}} \\ & \quad \left[F\left(-2i, -2j; \frac{1}{2}-i-j; \frac{3p-2}{4(p-1)}\right) - F\left(-(2i-1), -(2j+1); \frac{1}{2}-i-j; \frac{3p-2}{4(p-1)}\right) \right] \end{aligned}$$

Using Lemma 2.1 we get

$$\begin{aligned} & F\left(-2i, -2j; \frac{1}{2}-i-j; x\right) - F\left(-(2i-1), -(2j+1); \frac{1}{2}-i-j; x\right) \\ &= 2x \frac{(1-2i+2j)}{(1-2i-2j)} F\left(-2j, -2i+1, \frac{3}{2}-i-j, x\right), \end{aligned}$$

so that

$$\begin{aligned} & \mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_{2i}(u) & P_{2j+1}(u) \\ P_{2i-1}(u) & P_{2j}(u) \end{bmatrix} \\ &= \frac{\Gamma(i+j+\frac{1}{2})}{\sqrt{2\pi} \frac{(1-2i-2j)}{(1-2i+2j)} \left(-\frac{3p-2}{4(p-1)}\right)^{i+j}} \sqrt{\frac{3p-2}{p-1}} F\left(-2j, -2i+1, \frac{3}{2}-i-j, \frac{3p-2}{4(p-1)}\right). \end{aligned}$$

This proves (1).

Now we prove (2). Observe that by (5.7) we have

$$(5.9) \quad \mathbb{E}_{u \sim N(0, \sigma^2)} P_0(u) P_{2j}(u) e^{-\frac{u^2}{2}} = \frac{(-1)^j 4^{j+1} \Gamma(j + \frac{1}{2}) \left(\frac{p-1}{3p-2}\right)^{j+\frac{1}{2}}}{2\sqrt{2\pi}} = \frac{(-1) \Gamma(j + \frac{1}{2})}{2\sqrt{2\pi} \left(-\frac{3p-2}{4(p-1)}\right)^{j+1}} \sqrt{\frac{3p-2}{p-1}}.$$

Moreover, by Lemma 2.5 (2) we have

$$(5.10) \quad \begin{aligned} \mathbb{E}_{u \sim N(0, \sigma^2)} P_{-1}(u) P_{2j+1}(u) e^{-\frac{u^2}{2}} &= \frac{(-1)^{j+1} (2j+1)!}{2^j j!} \frac{(1-\sigma^2)^j \sigma^2}{\sqrt{1+\sigma^2}} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{\sigma^4}{\sigma^4-1}\right) \\ &= \frac{(-1)^{j+1} (2j+1)!}{2^j j!} \frac{\left(\frac{p-2}{2(p-1)}\right)^j \left(\frac{p}{2(p-1)}\right)}{\sqrt{\frac{3p-2}{2(p-1)}}} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right) \\ &= \frac{(-1)^{j+1} (2j+1)!}{2^{2j} \sqrt{2} j!} \frac{(p-2)^j p}{(p-1)^j (3p-2)} \sqrt{\frac{3p-2}{p-1}} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right). \end{aligned}$$

Combining (5.9) and (5.10) we see that $\mathbb{E}_{u \sim N(0, \sigma^2)} e^{-\frac{u^2}{2}} \det \begin{bmatrix} P_0(u) & P_{2j+1}(u) \\ P_{-1}(u) & P_{2j}(u) \end{bmatrix}$ equals

$$\sqrt{\frac{3p-2}{p-1}} \left[\frac{(-1)^j (2j+1)!}{2^{2j} \sqrt{2} j!} \frac{(p-2)^j p}{(p-1)^j (3p-2)} F\left(-j, \frac{1}{2}, \frac{3}{2}, \frac{-p^2}{(3p-2)(p-2)}\right) - \frac{\Gamma(j + \frac{1}{2})}{2\sqrt{2\pi} \left(-\frac{3p-2}{4(p-1)}\right)^{j+1}} \right],$$

which proves (2). \square

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APPENDIX A. SAGE CODE TO COMPUTE $E(n, p)$

Below we give the code for two SAGE scripts, that upon input m compute $E(2m+1, p)$ and $E(2m, p)$, respectively. Note that the one has to remove some line breaks to use the scripts.

A.1. The code for $E(2m+1, p)$.

```
#####
# The case n = 2m+1 is odd: Compute A and A_1, so that E(n,p)=1+A*A_1
# (A is the factor in front of the sum and A_1 is the sum)
#####
##### define the variables
p,m,n,i,j,x=var('p,m,n,i,j,x');
##### Set the value of m
m=2; n=2*m+1;
##### compute A
A(p)=sqrt(pi)*sqrt(p-1)^(n-2)*sqrt(3*p-2)/(prod(gamma(i/2) for i in (1..n)));
##### compute the determinants of the Gamma matrices
G_1 = matrix(m, lambda i,j: gamma(i+j+3/2));
G_1 = matrix(m, lambda i,j: det(G_1.matrix_from_rows_and_columns([0..i-1,i+1..m-1],[0..j-1,j+1..m-1]));
##### compute A_1
from sage.misc.mrange import cantor_product
L = list(cartesian_product_iterator([1..m], [1..m]));
A_1(p) = sum(G_1[i-1,j-1] * gamma(i+j-1/2) * ((1-2*i+2*j)/(3-2*i-2*j)) * ((-3*p-2)/(4*(p-1)))^(1-i-j)
            * hypergeometric([2-2*i,1-2*j],[5/2-i-j],(3*p-2)/(4*(p-1))) for (i,j) in L);
A_1(p) = A_1(p).simplify_hypergeometric();
##### compute E(n,p)
E_n_odd(p)=A(p)*A_1(p);
E_n_odd(p)=E_n_odd(p).factor()
E_n_odd(p)=E_n_odd(p)+1
print(E_n_odd(p)) # prints the formula (wrap 'latex()' around it to get tex code)
```

A.2. The code for $E(2m, p)$.

```
#####
# The case n = 2m is even: Compute B, B_1, B_2 and B_3, so that E(n,p)=B*(B_1-B_2+B_3)
# (B is the factor in front of the sum and B_1-B_2+B_3 is the sum)
#####
##### define the variables
p,m,n,i,j,x=var('p,m,n,i,j,x');
##### Set the value of m
m=2; n=2*m;
##### compute B
B(p)=sqrt(p-1)^(n-2)*sqrt(3*p-2)/(prod(gamma(i/2) for i in (1..n)));
##### compute the determinants of the Gamma matrices
G_2 = matrix(m, lambda i,j: gamma(i+j+1/2));
G_2 = matrix(m, lambda i,j: det(G_2.matrix_from_rows_and_columns([0..i-1,i+1..m-1],[0..j-1,j+1..m-1]));
##### compute B_1
B_1(p) = sum(sqrt(pi) * G_2[0,j] * (gamma(2*j+2)/((-1)^j*4^j*gamma(j+1))) * ((p-2)^j*p)/((p-1)^j*(3*p-2))
            * hypergeometric([-j, 1/2],[3/2],-p^2/((3*p-2)*(p-2))) for j in [0..m-1]);
B_1(p) = B_1(p).simplify_hypergeometric();
##### compute B_2
B_2(p) = sum(G_2[0,j] * gamma(j+1/2) * (-4*(p-1)/(3*p-2))^(j+1)/2 for j in [0..m-1]);
##### compute B_3
from sage.misc.mrange import cantor_product
L = list(cartesian_product_iterator([1..m-1], [0..m-1]));
B_3(p) = sum(G_2[i,j] * gamma(i+j+1/2) * ((1-2*i+2*j)/(1-2*i-2*j)) * ((-4*(p-1)/(3*p-2))^(i+j)
            * hypergeometric([-2*j,-2*i+1],[3/2-i-j],(3*p-2)/(4*(p-1))) for (i,j) in L);
B_3(p) = B_3(p).simplify_hypergeometric();
##### compute E(n,p)
E_n_even(p)=B(p) * (B_1(p)-B_2(p)+B_3(p));
E_n_even(p)=E_n_even(p).factor()
print(E_n_even(p)) # prints the formula (wrap 'latex()' around it to get tex code)
```